

Discussion 01

Weekly Material Recap and Example Problems

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1 General reminders

- **Assignment reminder:** Weekly homework is due on Fridays at 6:00 pm and must be **handwritten** (Ipad is ok, but solutions should not be typed).
- **GSI logistics:** My office hours are 3:30-5:00 pm Mondays and Thursdays in 347 Soda. Discussion section is 5:00-6:00 pm in 306 Soda.
- **Important reminder:** Any material found in the readings is fair game for exams. Although I will do my best to summarize the most important concepts here, I encourage you to keep up with the readings so that you are fully prepared.
- **Comment on LLMs:** LLMs have fundamentally changed the way we work and, consequently, the way we learn. I strongly encourage you to always find solutions yourself before consulting AI. AI is, of course, a powerful tool for accelerating your work, but it is important to build a foundation of fundamental knowledge so that you can eventually use the tool more effectively. In practice, we will always need to check the work of AI to ensure that our machines, buildings, and other systems are safe, which requires knowing how to perform engineering calculations ourselves. More generally, AI fundamentally cannot expand knowledge on its own, at least not reliably or to any great extent. Therefore, as a society, we must either continue to build our knowledge ourselves in order to expand the scientific canon or accept that things are as good as they are ever going to get. The second option is pretty depressing!

2 August 24–28: material recap

2.1 Brief math review

We start by reviewing a few simple math concepts that are central to what you have learned thus far.

Unit vectors

A unit vector is any vector of magnitude (or modulus) 1. That is, $\mathbf{u}$ is a unit vector if and only if

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = 1.$$

It follows that if $\mathbf{v}$ is any nonzero vector, then

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|} = |\mathbf{v}| \mathbf{e}_v, \quad \mathbf{e}_v = \frac{\mathbf{v}}{|\mathbf{v}|},$$

where $\mathbf{e}_v$ is the **unit vector in the direction of $\mathbf{v}$** . Dividing by the magnitude gives a vector of magnitude 1 while preserving its direction.



Same direction; unit magnitude after normalization.

Observe that $\mathbf{e}_v$ is unitless and has magnitude 1. The magnitude $|\mathbf{v}|$ carries the units of $\mathbf{v}$ (e.g., [m] for a position vector). Thus, in $\mathbf{v} = |\mathbf{v}|\mathbf{e}_v$, the scalar factor supplies the magnitude and units, while the unit vector supplies the direction.

To describe the orientation of a surface, we use a normal vector:

Normal vector to a plane

Consider a plane $\mathcal{S}$. A **unit normal vector** $\mathbf{n}$ is a vector of magnitude 1 perpendicular to $\mathcal{S}$. There are two possible choices, $\mathbf{n}$ and $-\mathbf{n}$; choosing one fixes the orientation of the plane. On the boundary of a body, we usually choose $\mathbf{n}$ to point outward.

For a smooth curved surface, the normal at a point is perpendicular to the tangent plane there.

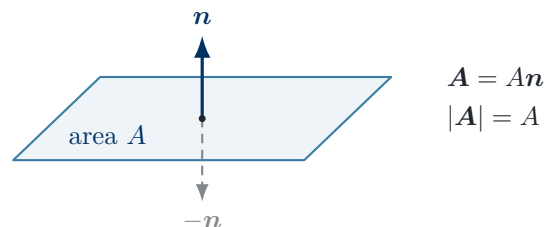
A general normal vector can encode both orientation and area:

Areal vector

An areal vector $\mathbf{A}$ [m²] denotes both the area and orientation of a flat surface patch. If the patch has a unit normal $\mathbf{n}$ and area A [m²], then

$$\mathbf{A} = A\mathbf{n} \quad [\text{m}^2].$$

For an infinitesimal patch, we write $d\mathbf{A} = \mathbf{n} dA$ [m²], where dA [m²] is its area.



Choosing $\mathbf{n}$ sets the orientation; multiplying by A [m²] sets the magnitude.

We can finally give a geometric definition of the cross product that lets you know what exactly we are computing:

Cross product

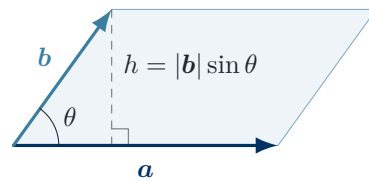
The cross product $\mathbf{a} \times \mathbf{b}$ is an algebraic operation between three-dimensional vectors that returns the areal vector associated with the parallelogram constructed from $\mathbf{a}$ and $\mathbf{b}$. Its orientation is determined by their order through the **right-hand rule**. In particular,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\theta)\mathbf{n},$$

where $\theta \in [0, \pi]$ [rad] is the angle between $\mathbf{a}$ and $\mathbf{b}$. For nonparallel vectors, $\mathbf{n}$ is the unit normal obtained by curling the fingers of your right hand from $\mathbf{a}$ toward $\mathbf{b}$ through this angle; your thumb

points along $\mathbf{n}$. Reversing the order reverses the normal, so

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$



$$\text{area} = |\mathbf{a}|h = |\mathbf{a}||\mathbf{b}| \sin \theta$$

$\odot$ $\mathbf{a} \times \mathbf{b}$
out of the page

$\otimes$ $\mathbf{b} \times \mathbf{a}$
into the page

The magnitude is the parallelogram's area [m^2]; the direction is perpendicular to its plane.

Common pitfall

If $\mathbf{a}$ and $\mathbf{b}$ are parallel (including opposite directions), or either vector is zero, the parallelogram has zero area. Therefore, $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ in these cases!

In other words, the cross product is one way of multiplying two vectors to obtain another vector. Although, in principle, the cross product can always be computed from this definition alone, the following identities are useful for a right-handed Cartesian coordinate system $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$:

- $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$.
- $\mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x$.
- $\mathbf{e}_z \times \mathbf{e}_x = \mathbf{e}_y$. $\leftarrow$ note this ordering!

Reversing any of these products changes its sign, and $\mathbf{e}_i \times \mathbf{e}_i = \mathbf{0}$. The cross product is also linear in each argument. For example:

$$\mathbf{a} \times (\alpha \mathbf{b} + \gamma \mathbf{c}) = \alpha(\mathbf{a} \times \mathbf{b}) + \gamma(\mathbf{a} \times \mathbf{c}).$$

2.2 Main ideas

Consider a general body $\mathcal{B}$, acted on by a collection of n external forces $\{\mathbf{F}_i\}$ [N]. In general, the sum of the external forces [N] on a body is equal to the time rate of change of its momentum, $\mathbf{p}$ [kg m/s]. Likewise, the sum of the external moments [N m] about a fixed point p in an inertial reference frame is equal to the time rate of change of the angular momentum about that point, $\mathbf{L}_p$ [kg m²/s]. However, this course primarily deals with statics, meaning that bodies remain at rest in an inertial reference frame and hence $\frac{d}{dt}\mathbf{p} = \mathbf{0}$ [N] and $\frac{d}{dt}\mathbf{L}_p = \mathbf{0}$ [N m]. These facts lead to the following equations, which will be utilized throughout the course:

Force and moment equilibrium

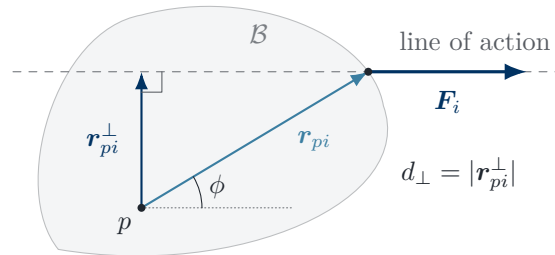
- **Force equilibrium:** $\sum_{i=1}^n \mathbf{F}_i = \mathbf{0}$ [N] $\implies \sum_{i=1}^n \mathbf{F}_i \cdot \mathbf{e}_j = 0$ [N] for $j = x, y$, or z .
- **Moment equilibrium:** $\sum_{i=1}^n \mathbf{r}_{pi} \times \mathbf{F}_i = \mathbf{0}$ [N m] $\implies \sum_{i=1}^n (\mathbf{r}_{pi} \times \mathbf{F}_i) \cdot \mathbf{e}_j = 0$ [N m] for $j = x, y$, or z .

From the component expressions, this supplies 3 equations in 2D (where forces [N] act only in a single plane) or 6 equations in 3D.

Here, $\mathbf{r}_{pi}$ [m] is the **position vector** from p to the point where $\mathbf{F}_i$ [N] is applied. It need not be perpendicular to the force [N]. We may also choose any other point on the force's **line of action** without changing its moment [N m] about p .

The perpendicular distance $d_\perp$ [m] from p to this line is the **moment arm** (or **lever arm**). If ϕ [rad] is the angle between $\mathbf{r}_{pi}$ [m] and $\mathbf{F}_i$ [N], the moment magnitude [N m] is

$$|\mathbf{M}_p| = |\mathbf{r}_{pi} \times \mathbf{F}_i| = |\mathbf{r}_{pi}| |\mathbf{F}_i| \sin \phi = d_\perp |\mathbf{F}_i| \quad [\text{N m}].$$



Both $\mathbf{r}_{pi}$ [m] and $\mathbf{r}_{pi}^\perp$ [m] give the same moment [N m] about p ; the latter is perpendicular to the force [N].

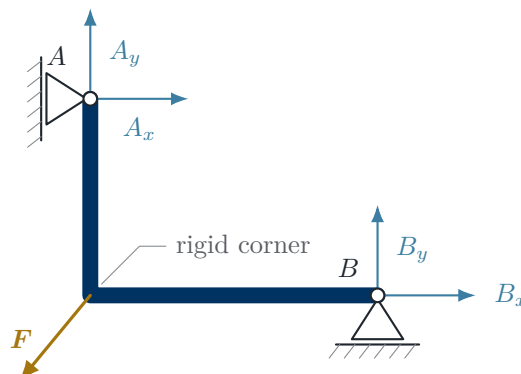
If an external couple moment [N m] is specified separately (a moment [N m] produced by equal and opposite forces [N] whose lines of action are separated by a perpendicular distance h [m]), it must also be added to the moment equilibrium equation.

3 Static indeterminacy

It will sometimes be the case that there are more unknown forces [N] than there are independent equilibrium equations to solve for them. In this case, we cannot solve for all of the forces [N] from equilibrium alone, and must also account for deformation compatibility and constitutive response (e.g., how the material deforms under load). This is a topic for a later date, but it is important to know when equilibrium alone is sufficient to solve a problem.

- In 2D, a single body provides up to 3 independent equilibrium equations, while in 3D it provides up to 6. For a stable system, the excess of unknown reaction components [N] over the number of independent equilibrium equations available to determine them is the **degree of external static indeterminacy**.

For example, consider the L-shaped bracket below, with a rigid corner, pinned supports at A and B , and a known force $\mathbf{F}$ [N] acting at the corner. Each pin contributes two unknown reaction components: A_x [N], A_y [N], B_x [N], and B_y [N]. There are 4 unknowns but only 3 independent equilibrium equations, so the bracket is **externally statically indeterminate to degree 1**.



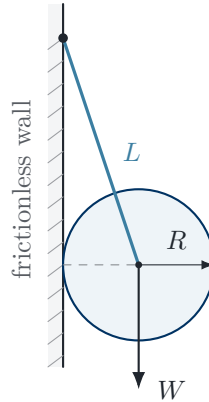
Two pins give four unknown reaction components [N]; planar equilibrium gives three equations.

4 Example problems

Although the problems are symbolic, report the answers with the appropriate units.

Example 1: Hanging weight

Consider a uniform sphere of radius R [cm] and weight W [N] resting against a frictionless vertical wall. A massless wire of length L [cm] connects the sphere's center to a point on the wall above it, as shown. What is the tension [N] in the wire in terms of W , R , and L ?



Solution

The sphere's center is a horizontal distance R [cm] from the wall. Since the wire has length L [cm], the vertical distance between its endpoints is

$$h = \sqrt{L^2 - R^2} \quad [\text{cm}].$$

The vertical component of the tension T [N] is therefore Th/L [N].

The frictionless wall exerts only a horizontal reaction [N]. Vertical force equilibrium gives

$$T \frac{\sqrt{L^2 - R^2}}{L} - W = 0 \quad [\text{N}] \quad \Rightarrow \quad \boxed{T = \frac{WL}{\sqrt{L^2 - R^2}} \quad [\text{N}].}$$

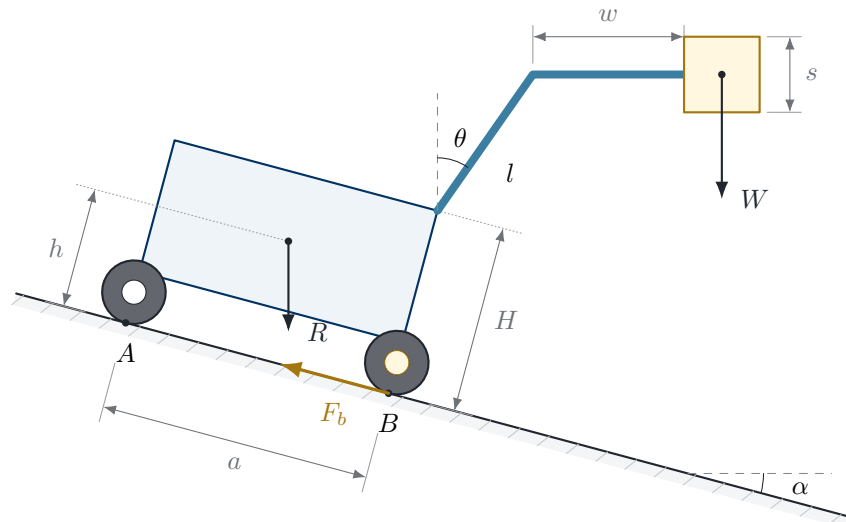
Example 2: Robot on an incline

A robot of weight R [N] carries a uniform square object of weight W [N] and side length s [cm]. The road slopes downward toward the object at an angle α [°] to the horizontal. The wheel contact points, A and B , are a distance a [cm] apart along the road.

The robot's center of mass is a distance h [cm] above the midpoint of AB , measured perpendicular to the road. The arm's base is a perpendicular distance H [cm] above B . An arm of length l [cm] makes an angle θ [°] with the **true vertical**; a second arm of length w [cm] is **horizontal** and attaches to the midpoint of the object's left face. Treat the arms as rigid and massless, with fixed connections.

The downhill wheel B is locked by a brake, while the uphill wheel A rolls freely. The ground supplies an uphill static-friction force F_b [N] at B . Assume sufficient traction and braking capacity to prevent slipping or rolling, and assume the unloaded robot is stable.

1. Find the braking force F_b [N] required to hold the robot at rest for a given object weight W [N].
2. Find the maximum weight W [N] that the robot can carry before tipping over.



Solution

Let N_A [N] and N_B [N] be the normal reactions at the two wheel contacts. Resolving forces parallel and perpendicular to the road gives

$$\begin{aligned} F_b &= (R + W) \sin \alpha \quad [\text{N}], \\ N_A + N_B &= (R + W) \cos \alpha \quad [\text{N}]. \end{aligned}$$

The braking force [N] is the external static-friction force [N] at the ground. The brake's internal forces [N] and torques [N cm] are not added separately to the free-body diagram of the complete robot and object.

To find the tipping limit, take moments [N cm] about the downhill contact point B . Both F_b [N] and N_B [N] act through B , so neither contributes a moment [N cm] about that point.

The horizontal distance [cm] from B to the robot's center of mass, measured toward the uphill side, is

$$d_R = \frac{a}{2} \cos \alpha - h \sin \alpha \quad [\text{cm}].$$

The object's center lies a horizontal distance [cm]

$$d_W = H \sin \alpha + l \sin \theta + w + \frac{s}{2} \quad [\text{cm}]$$

to the downhill side of B . Notice that θ [°] is measured from the true vertical, so the first arm's horizontal projection is still $l \sin \theta$ [cm].

Moment equilibrium about B , with counterclockwise positive, gives

$$-aN_A + Rd_R - Wd_W = 0 \quad [\text{N cm}], \quad N_A = \frac{Rd_R - Wd_W}{a} \quad [\text{N}].$$

At the onset of tipping, the uphill wheel loses contact and $N_A = 0$ [N]. Therefore,

$$W_{\max} = \frac{R \left(\frac{a}{2} \cos \alpha - h \sin \alpha \right)}{H \sin \alpha + l \sin \theta + w + \frac{s}{2}} \quad [\text{N}].$$

The corresponding braking force [N] is

$$F_{b,\max} = (R + W_{\max}) \sin \alpha \quad [\text{N}].$$

A positive payload capacity [N] requires $\frac{a}{2} \cos \alpha > h \sin \alpha$ [cm], consistent with the assumption that the unloaded robot is stable. Setting $\alpha = 0^\circ$ recovers the level-ground tipping result.

5 Exercise

Although the problem is symbolic, report the answer with the appropriate units. For numerical substitution in the formulas, convert lengths to meters and forces to newtons: $1\ \mu\text{m} = 10^{-6}\text{ m}$ and $1\ \text{pN} = 10^{-12}\text{ N}$. Report particle sizes in micrometers and forces in piconewtons.

Example 3: Battery electrode microstructure

A thin battery electrode is held together using a polymeric carbon binder. Microscopic channels can be ablated into the electrode material to improve effective electrochemical properties, leaving narrow, pillar-like structures. Consider the following simplified model of one such structure.

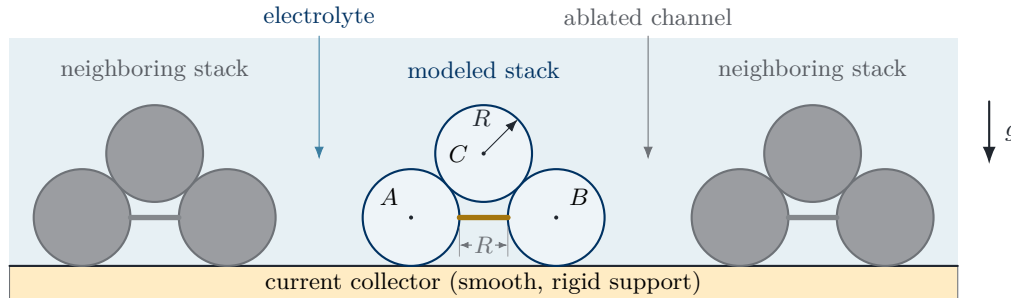
The pillar consists of three parallel rows of identical spherical particles, with two rows below and one above. End confinement keeps the rows aligned, and we assume that each group of three particles behaves identically. We can therefore analyze the representative group shown below.

Each particle has radius $R\ [\mu\text{m}]$ and mass density $\rho\ [\text{kg}/\text{m}^3]$. The two lower particles rest on a smooth, rigid horizontal surface. The binder is idealized as a massless horizontal cable joining their facing surfaces at the height of their centers. The cable has length $R\ [\mu\text{m}]$, so the lower centers are $3R\ [\mu\text{m}]$ apart. The upper particle rests freely on the lower two. Neglect friction at all contacts.

Taking the gravitational acceleration to be $g\ [\text{m}/\text{s}^2]$, each particle has weight $[\text{pN}]$

$$W = \frac{4}{3}\pi R^3 \rho g \quad [\text{pN}].$$

If the cable can support a maximum tension $F\ [\text{pN}]$ before risking damage to the binder, **what is the maximum allowable particle radius $R\ [\mu\text{m}]$?**



The gold segment represents the binder cable in the modeled stack. For this mechanical model, neglect buoyancy $[\text{pN}]$ and other forces $[\text{pN}]$ exerted by the electrolyte.

Solution

Units: In the numerical expressions below, use lengths in $[\mu\text{m}]$, forces in $[\text{pN}]$, mass density ρ in $[\text{kg}/\text{m}^3]$, and gravitational acceleration g in $[\text{m}/\text{s}^2]$. The product ρg therefore initially has units $[\text{N}/\text{m}^3]$. The required conversion is

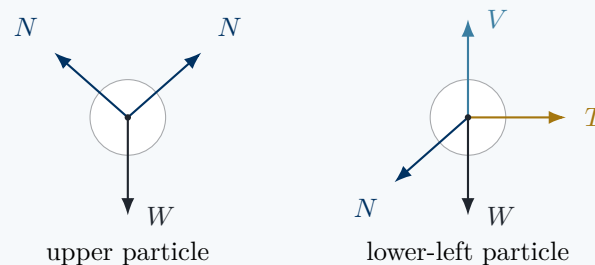
$$1 \text{ N}/\text{m}^3 = \frac{10^{12} \text{ pN}}{10^{18} \mu\text{m}^3} = 10^{-6} \text{ pN}/\mu\text{m}^3.$$

We will carry this conversion through explicitly.

Let N $[\text{pN}]$ be the magnitude of either contact force on the upper particle, and let T $[\text{pN}]$ be the cable tension. Since the contacts are frictionless, each contact force $[\text{pN}]$ acts along the line joining the particle centers.

The upper center is $2R$ $[\mu\text{m}]$ from either lower center. Its horizontal separation from either lower center is $3R/2$ $[\mu\text{m}]$. Thus, if θ $^\circ$ is the angle of a contact normal above the horizontal,

$$\cos \theta = \frac{3R/2}{2R} = \frac{3}{4}, \quad \sin \theta = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}.$$



By symmetry, the two contact forces $[\text{pN}]$ on the upper particle have equal magnitude. Its vertical equilibrium gives

$$2N \sin \theta = W \quad \Rightarrow \quad N = \frac{2W}{\sqrt{7}} \quad [\text{pN}].$$

Using the numerical values in the stated units, each particle's weight is

$$W = 10^{-6} \frac{4}{3} \pi R^3 \rho g \quad [\text{pN}].$$

Horizontal equilibrium of either lower particle then gives

$$T = N \cos \theta = \frac{3W}{2\sqrt{7}} = 10^{-6} \frac{2\pi \rho g}{\sqrt{7}} R^3 \quad [\text{pN}].$$

The lower particles' own weights $[\text{pN}]$ affect the vertical support reactions $[\text{pN}]$, not the cable tension $[\text{pN}]$. In fact,

$$V = W + N \sin \theta = \frac{3W}{2} \quad [\text{pN}]$$

for each lower particle. The limiting size occurs at $T = F$ $[\text{pN}]$, giving

$$R_{\max}^3 = 10^6 \frac{\sqrt{7} F}{2\pi \rho g} \quad [\mu\text{m}^3].$$

Taking the cube root,

$$R_{\max} = 10^2 \left(\frac{\sqrt{7} F}{2\pi \rho g} \right)^{1/3} \quad [\mu\text{m}].$$

Here, substitute the numerical values of F in $[\text{pN}]$, ρ in $[\text{kg}/\text{m}^3]$, and g in $[\text{m}/\text{s}^2]$.

The corresponding maximum diameter is $2R_{\max}$ $[\mu\text{m}]$. For this geometrically similar family of arrangements, the required binder force $[\text{pN}]$ scales as R^3 $[\mu\text{m}^3]$.