

Discussion 02

Weekly Material Recap and Example Problems

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1 General reminders

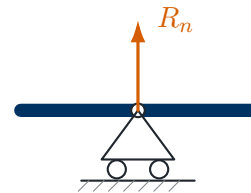
- **Assignment reminder:** Weekly homework is due on Fridays at 6:00 pm and must be **handwritten**. Last week's homework looked pretty good, but let me know (in section or by email) if there are major questions.
- **GSI logistics:** My office hours are 3:30-5:00 pm Mondays and Thursdays in 347 Soda. Discussion section is 5:00-6:00 pm in 306 Soda.

2 August 31–September 4: material recap

2.1 Canonical support reactions

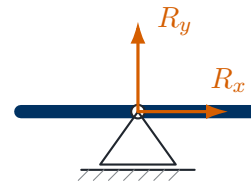
Roller:

Prevents translation normal to the support surface, but allows translation parallel to the surface and rotation at the support.



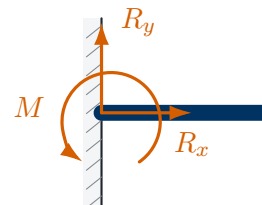
Pin:

Prevents all movement in the horizontal and vertical directions, but allows for rotation about the connection point.



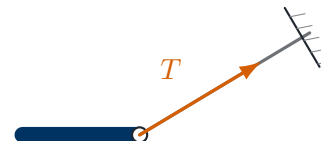
Fixed/built-in connection:

Resists vertical and horizontal translation, as well as rotation about the connection point. Provides a reaction couple moment.



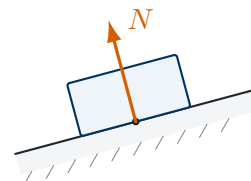
Tension cable:

Can exert only a tensile force directed along the cable; only resists motion away from the cable's anchor.



Frictionless smooth support:

Only resists motion into the surface; provides only an **outward** normal reaction force.



2.2 Force systems

The concept of a force system formalizes the idea of a resultant force and its associated moment about any point p in space.

Force system

Recall the definitions of the

- **resultant force:** $\sum_{i=1}^n \mathbf{F}_i = \mathbf{R}$ [N],
- **resultant moment:** $\sum_{i=1}^n \mathbf{r}_{pi} \times \mathbf{F}_i = \mathbf{M}_R^{(p)}$ [N m].

Any separately applied couple moments must also be included in $\mathbf{M}_R^{(p)}$.
The force system at point p is denoted as

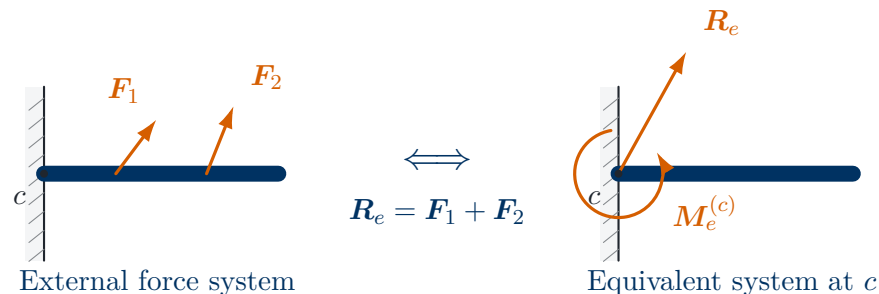
$$\{\mathbf{R}, \mathbf{M}_R^{(p)}\}.$$

In this course, we primarily care about cases of equilibrium where, if all forces and moments are accounted for,

$$\{\mathbf{R}, \mathbf{M}_R^{(p)}\} = \{\mathbf{0}, \mathbf{0}\}.$$

Other cases to note are

- $\{\mathbf{R}, \mathbf{M}_R^{(p)}\} = \{\mathbf{R}, \mathbf{0}\}$: a point force system (the force system reduces to a single resultant force whose line of action passes through p).
- $\{\mathbf{R}, \mathbf{M}_R^{(p)}\} = \{\mathbf{0}, \mathbf{M}_R^{(p)}\}$: a couple force system (equal and opposite forces acting with some offset distance between them).



It can be useful, however, to consider the partial force system associated only with the externally applied forces to calculate, for example, the reactions at a fixed connection. If $\mathbf{R}_e$ denotes the resultant of the externally applied forces, excluding the support reactions, and $\mathbf{M}_e^{(c)}$ denotes their resultant moment about the connection point c , then

$$\{-\mathbf{R}_e, -\mathbf{M}_e^{(c)}\}$$

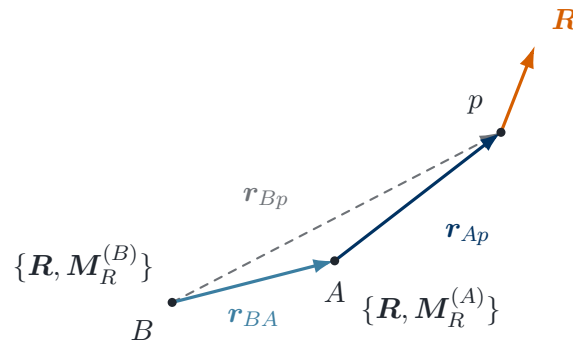
characterizes the reactions at a fixed connection exactly. These are the force and moment that must be added at c to obtain an equilibrium system. The figure demonstrates how to construct a useful **equivalent force system**: a system represented entirely at point c that has the same resultant force and resultant moment about every point in space as the actual applied forces.

2.3 Change of reference point

The resultant force $\mathbf{R}$ does not depend on the reference point, but the resultant moment often does. Suppose that a force system is known at point A :

$$\{\mathbf{R}, \mathbf{M}_R^{(A)}\}.$$

We wish to express the same force system at another point B .



For each applied force $\mathbf{F}_i$,

$$\mathbf{r}_{Bi} = \mathbf{r}_{BA} + \mathbf{r}_{Ai}, \quad \mathbf{r}_{BA} = \mathbf{x}_A - \mathbf{x}_B.$$

Therefore,

$$\begin{aligned} \mathbf{M}_R^{(B)} &= \sum_{i=1}^n \mathbf{r}_{Bi} \times \mathbf{F}_i = \sum_{i=1}^n (\mathbf{r}_{BA} + \mathbf{r}_{Ai}) \times \mathbf{F}_i = \mathbf{r}_{BA} \times \sum_{i=1}^n \mathbf{F}_i + \sum_{i=1}^n \mathbf{r}_{Ai} \times \mathbf{F}_i \\ &= \mathbf{r}_{BA} \times \mathbf{R} + \mathbf{M}_R^{(A)} \end{aligned}$$

Change of reference point

The same force system expressed at A and B satisfies

$$\boxed{\mathbf{M}_R^{(B)} = \mathbf{M}_R^{(A)} + \mathbf{r}_{BA} \times \mathbf{R}}, \quad \mathbf{r}_{BA} = \mathbf{x}_A - \mathbf{x}_B.$$

Thus,

$$\{\mathbf{R}, \mathbf{M}_R^{(A)}\} \longrightarrow \{\mathbf{R}, \mathbf{M}_R^{(B)}\}.$$

The resultant force remains unchanged; only its associated moment changes.

Common pitfall

The vector $\mathbf{r}_{BA} = \mathbf{x}_A - \mathbf{x}_B$ points from B to A . It represents the separation between point B and the force system represented at A , as though its resultant force acted there. Reversing this vector incorrectly reverses the sign of the additional moment.

Note that to sum two force systems, they must first be defined relative to the same point. Therefore the sum of a force system defined at A and a force system defined at B is

$$\{\mathbf{R}_1, \mathbf{M}_{R_1}^{(A)}\}_A + \{\mathbf{R}_2, \mathbf{M}_{R_2}^{(B)}\}_B = \{\mathbf{R}_1 + \mathbf{R}_2, \mathbf{r}_{BA} \times \mathbf{R}_1 + \mathbf{M}_{R_1}^{(A)} + \mathbf{M}_{R_2}^{(B)}\}_B.$$

This system can then be moved to another point C by the usual change of reference point formula, if desired.

3 Example problems

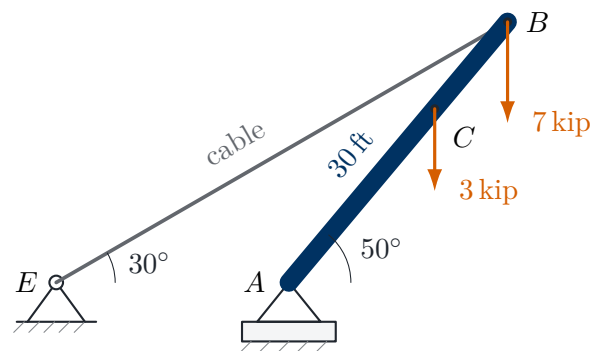
Report the answers with the appropriate units.

Example 1: Equilibrium of a crane arm

A rigid crane arm AB of length 30 ft is connected at its base A to a massive foundation by a pin support. The arm is inclined at 50° above the horizontal. A cable attached at the top of the arm at B is anchored to the ground at E and makes an angle of 30° with the horizontal. A downward load of 3 kip acts at point C , located two-thirds of the distance from A to B :

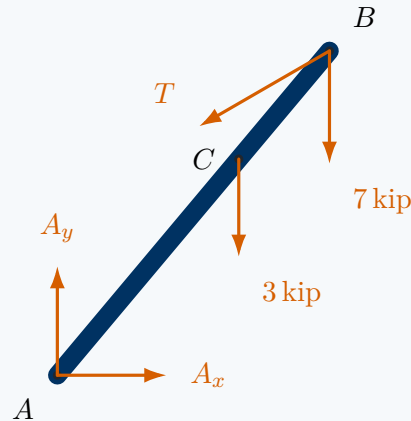
$$AC = \frac{2}{3}AB = 20 \text{ ft.}$$

A downward load of 7 kip acts at the top of the arm at B . Determine the tension in the cable and the horizontal and vertical support forces at A .



Solution

Isolating the crane arm gives the following free-body diagram:



The cable is a two-force member in tension with components

$$T_x = -T \cos(30^\circ), \quad T_y = -T \sin(30^\circ).$$

Taking moments about A and using $M_A = xF_y - yF_x$ gives

$$\begin{aligned} 0 &= (30 \cos 50^\circ)T_y - (30 \sin 50^\circ)T_x - 7(30 \cos 50^\circ) - 3(20 \cos 50^\circ) \\ &= -30T \cos 50^\circ \sin 30^\circ + 30T \sin 50^\circ \cos 30^\circ - 270 \cos 50^\circ. \end{aligned}$$

Thus,

$$T = \frac{9 \cos(50^\circ)}{\sin(20^\circ)} \approx 16.9 \text{ kip}.$$

Force equilibrium gives

$$\begin{aligned} \sum F_x = 0 : \quad & A_x - T \cos 30^\circ = 0, \\ \sum F_y = 0 : \quad & A_y - T \sin 30^\circ - 3 - 7 = 0. \end{aligned}$$

Therefore,

$$A_x \approx 14.6 \text{ kip}, \quad A_y \approx 18.5 \text{ kip}.$$

Example 2: Equivalent point-force system

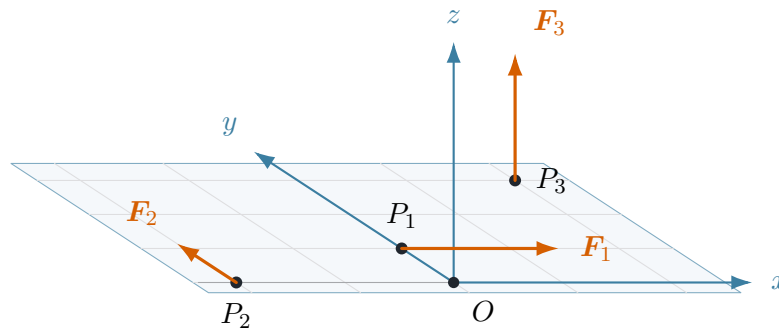
Three forces act at the points P_i shown below. Their position vectors relative to the origin O and their force vectors are

i	$\mathbf{x}_i$ [m]	$\mathbf{F}_i$ [kN]
1	$(0, 1, 0)$	$2\mathbf{e}_x$
2	$(-2, 0, 0)$	$\mathbf{e}_y$
3	$(2, 3, 0)$	$2\mathbf{e}_z$

Let

$$\mathbf{R} = \sum_{i=1}^3 \mathbf{F}_i, \quad \mathbf{M}_R^{(O)} = \sum_{i=1}^3 \mathbf{x}_i \times \mathbf{F}_i.$$

1. Calculate $\mathbf{R}$ and $\mathbf{M}_R^{(O)}$, and verify that $\mathbf{R} \cdot \mathbf{M}_R^{(O)} = 0$.
2. Find the coordinates of the point $P = (x, y, 0)$ in the xy -plane such that the force system can be represented at P as $\{\mathbf{R}, \mathbf{0}\}_P$.



Solution

The resultant force is

$$\begin{aligned}\mathbf{R} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= (2, 0, 0) + (0, 1, 0) + (0, 0, 2) \\ &= (2, 1, 2) \quad [\text{kN}].\end{aligned}$$

The moments of the three forces about O are

$$\begin{aligned}\mathbf{x}_1 \times \mathbf{F}_1 &= (0, 1, 0) \times (2, 0, 0) = (0, 0, -2), \\ \mathbf{x}_2 \times \mathbf{F}_2 &= (-2, 0, 0) \times (0, 1, 0) = (0, 0, -2), \\ \mathbf{x}_3 \times \mathbf{F}_3 &= (2, 3, 0) \times (0, 0, 2) = (6, -4, 0).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{M}_R^{(O)} &= \sum_{i=1}^3 \mathbf{x}_i \times \mathbf{F}_i \\ &= (0, 0, -2) + (0, 0, -2) + (6, -4, 0) \\ &= (6, -4, -4) \quad [\text{kN m}].\end{aligned}$$

The required orthogonality condition is satisfied:

$$\mathbf{R} \cdot \mathbf{M}_R^{(O)} = (2, 1, 2) \cdot (6, -4, -4) = 12 - 4 - 8 = 0.$$

Now let

$$\mathbf{x}_P = (x, y, 0) \quad [\text{m}].$$

Because $\mathbf{x}_O = \mathbf{0}$,

$$\mathbf{r}_{PO} = \mathbf{x}_O - \mathbf{x}_P = (-x, -y, 0) \quad [\text{m}].$$

Changing the reference point from O to P gives

$$\begin{aligned}\mathbf{M}_R^{(P)} &= \mathbf{M}_R^{(O)} + \mathbf{r}_{PO} \times \mathbf{R} \\ &= (6, -4, -4) + (-x, -y, 0) \times (2, 1, 2) \\ &= (6, -4, -4) + (-2y, 2x, -x + 2y) \\ &= (6 - 2y, -4 + 2x, -4 - x + 2y).\end{aligned}$$

For the force system to reduce to $\{\mathbf{R}, \mathbf{0}\}_P$, we require

$$\mathbf{M}_R^{(P)} = \mathbf{0}.$$

Thus,

$$6 - 2y = 0, \quad -4 + 2x = 0, \quad -4 - x + 2y = 0.$$

The first two equations give

$$y = 3, \quad x = 2.$$

These values also satisfy the third equation:

$$-4 - 2 + 2(3) = 0.$$

Therefore, the required point in the xy -plane is

$$\boxed{P = (2, 3, 0) \text{ m.}}$$

At this point, the equivalent force system is

$$\boxed{\{\mathbf{R}, \mathbf{0}\}_P = \{(2, 1, 2) \text{ kN}, \mathbf{0}\}_P.}$$

4 Challenge exercise

Example 3: Support reactions in a rigid frame

Consider the single rigid frame $ABCD$ shown below. The frame is supported by a roller at A on a horizontal smooth surface and by a pin at D .

$$AB = BC = ED = a, \quad CD = a + b.$$

A force of magnitude F acts at C horizontally to the left. A second force of the same magnitude acts at B , upward and to the right at an angle θ above the horizontal.

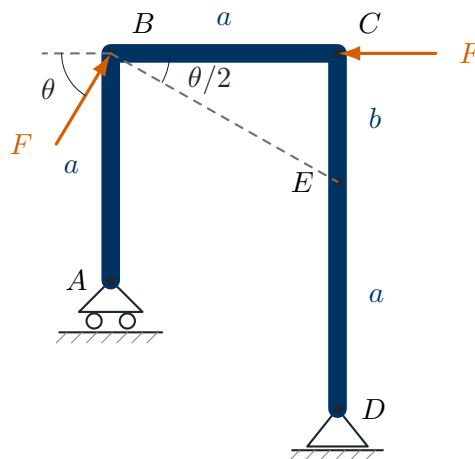
A ray drawn from B at an angle $\theta/2$ below the horizontal intersects the right member at E , located a distance b below C .

1. Determine the roller reaction at A and the horizontal and vertical pin reactions at D in terms of F , a , and b .

Hint: use the half/double angle formulas for sin, which can be written as

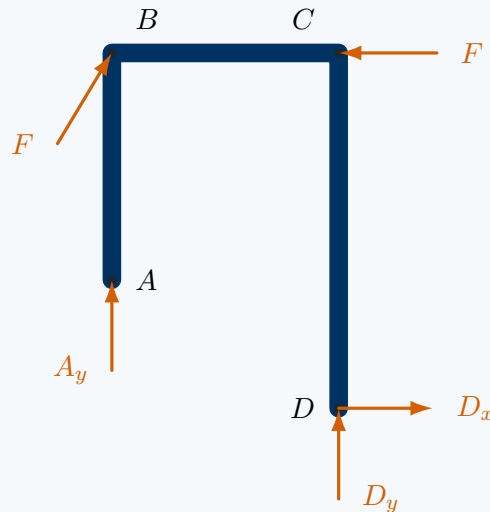
$$2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta \quad \text{and} \quad \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}, \quad \text{respectively.}$$

2. Determine the positive ratio a/b for which the reaction at the roller vanishes with $F \neq 0$.



Solution

The free-body diagram of the entire rigid frame is



From the geometry, we have $\tan \frac{\theta}{2} = \frac{b}{a}$, and so $\sin \frac{\theta}{2} = \frac{b}{\sqrt{a^2+b^2}}$ and $\cos \frac{\theta}{2} = \frac{a}{\sqrt{a^2+b^2}}$. From the hint, it then follows that

$$\sin \theta = \frac{2ab}{a^2 + b^2}, \quad 1 - \cos \theta = \frac{2b^2}{a^2 + b^2}, \quad \frac{\sin \theta}{1 - \cos \theta} = \frac{a}{b}.$$

Taking moments about B gives

$$(a + b)D_x + aD_y = 0 \implies D_y = -\frac{a + b}{a}D_x.$$

Horizontal force equilibrium then gives

$$D_x + F \cos \theta - F = 0 \implies D_x = F(1 - \cos \theta).$$

Finally, vertical force equilibrium gives

$$A_y + D_y + F \sin \theta = 0 \implies A_y = F \left[\frac{a + b}{a}(1 - \cos \theta) - \sin \theta \right] = F(1 - \cos \theta) \left[\frac{a + b}{a} - \frac{\sin \theta}{1 - \cos \theta} \right].$$

Plugging in our trigonometric identities, we are left with

$$\boxed{D_x = \frac{2Fb^2}{a^2 + b^2}, \quad D_y = -\frac{2Fb^2}{a^2 + b^2} \frac{a + b}{a}, \quad A_y = -\frac{2Fb^2}{a^2 + b^2} \left[\frac{a}{b} - \frac{a + b}{a} \right]},$$

or, letting $\varphi = \frac{a}{b}$, these can be written as

$$D_x = \frac{2F}{\varphi^2 + 1}, \quad D_y = -\frac{2F}{\varphi^2 + 1} \frac{\varphi + 1}{\varphi}, \quad A_y = -\frac{2F}{\varphi^2 + 1} \frac{\varphi^2 - \varphi - 1}{\varphi}.$$

For nonzero F , $A_y = 0$ when $\varphi^2 - \varphi - 1 = 0$. Taking the positive solution via the quadratic formula, we are left with

$$\boxed{\varphi = \frac{a}{b} = \frac{1 + \sqrt{5}}{2}},$$

which is the golden ratio.