

## Discussion 04

Weekly Material Recap and Example Problems

**Instructor:** Tim Carlson

**Contact:** [timothy.carlson@berkeley.edu](mailto:timothy.carlson@berkeley.edu)

**Date:** September 24, 2026

**Version:** Instructor copy

### 1 General reminders

- **Material coverage:** We are now moving from statics to mechanics of materials, meaning that the reference text is now [G]: "Engineering Mechanics of Deformable Solids" by Professor Govindjee. Remember there is a problem from this book in the HW this week!
- **Midterm 1:** The first midterm went fairly well. Remember: for many questions, writing what you know may be sufficient to get *some* points. Moment calculation was a serious issue for many.
- **GSI logistics reminder:** My office hours are 3:30-5:00 pm Mondays and Thursdays in 347 Soda. Discussion section is 5:00-6:00 pm in 306 Soda.

### 2 September 14–September 18: material recap

#### 2.1 Friction

When two surfaces that are not perfectly smooth are in contact, the contact force can be resolved into a normal force  $N$  and a tangential friction force  $F_f$ . Friction opposes relative sliding, or the tendency to slide, along the contact surface.

##### Coulomb friction

For surfaces that are not sliding relative to one another, the friction force adjusts to maintain equilibrium, provided

$$|F_f| \leq \mu_s N.$$

with equality at impending sliding. Once the surfaces are sliding, the kinetic-friction model gives

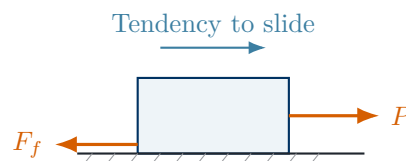
$$|F_f| = \mu_k N,$$

with friction directed opposite the relative sliding velocity. If  $\phi$  is the angle the resultant contact force makes with the contact normal, then

$$\tan \phi = \frac{|F_f|}{N} \leq \mu_s$$

For surfaces that are not sliding. Observe that the friction coefficients are dimensionless!

**Static friction is not automatically equal to  $\mu_s N$ .** First determine the friction needed for equilibrium, then check whether the contact can supply it.



## 2.2 Introduction to strain, 1-D deformation

Consider a straight bar undergoing only axial displacement. Let  $x$  denote a point's position before deformation, and let  $u(x)$  be its displacement along the bar. Its new position is  $x + u(x)$ .

Two points initially at  $x$  and  $x + \Delta x$  have a deformed separation

$$\Delta x \mapsto \Delta x + u(x + \Delta x) - u(x).$$

The change in separation, divided by the original separation, is therefore

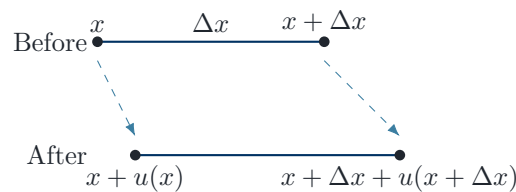
$$\frac{u(x + \Delta x) - u(x)}{\Delta x}.$$

### Axial strain

Taking the limit as the points approach one another gives the local axial strain:

$$\varepsilon(x) = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} = \frac{du}{dx}.$$

Positive strain indicates extension; negative strain indicates shortening. Strain is dimensionless.



For a finite interval  $[a, b]$ , the average axial strain is

$$\varepsilon_{\text{avg}} = \frac{u(b) - u(a)}{b - a}.$$

For a bar (or region of a bar) subject to uniform linear strain, the average strain is the strain everywhere.

Note that there is a way to connect strain to the idea of force distributions we learned previously. Let  $L$  and  $\hat{L}$  denote the initial and *deformed* length of a bar, respectively. The total change in length is

$$\hat{L} - L = u(L) - u(0) = \int_0^L \frac{du}{dx} dx = \int_0^L \varepsilon(x) dx.$$

Hence, strain in this context can be equivalently viewed as a distributed deformation over a length (or, in mathematical terms, a displacement density).

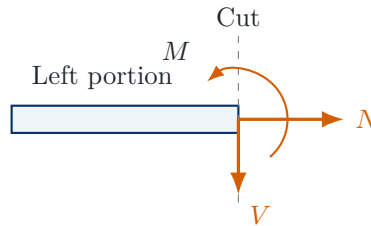
## 2.3 Internal force and moment diagrams

External forces tell us how a whole body is loaded. To determine the *internal* forces on a particular section, make an imaginary cut and draw a free body diagram of either side.

For a planar beam, the internal forces can be replaced by the following three resultants:

- $N$ : axial force, parallel to the beam;
- $V$ : shear force, transverse to the beam;
- $M$ : bending moment.

We use the following positive directions on the cut face of the **left portion**:  $N$  to the right,  $V$  downward, and  $M$  counterclockwise. The forces and moment on the opposite portion are equal and opposite.



Positive internal resultants; external loads are omitted here.

### Cut, isolate, and balance

Find the support reactions from the whole-body free body diagram. Then cut at a position  $l$  and apply equilibrium to the isolated left portion, taking moments about the cut:

$$\sum F_x^{\text{external}} + N(l) = 0, \quad \sum F_y^{\text{external}} - V(l) = 0, \quad \sum M^{\text{external}} + M(l) = 0.$$

Repeat in each region separated by a concentrated load, applied couple, or change in distributed loading. The resulting functions  $N(l)$ ,  $V(l)$ , and  $M(l)$  are plotted along the member.

A negative answer means the actual resultant acts opposite to its indicated positive direction. **Always keep the same convention throughout the calculation and for internal force diagrams.**

## 3 Example problems

Report answers with appropriate units. Draw a free body diagram where applicable.

### Example 1: Displacement and local strain

A straight bar of original length  $L = 2$  m undergoes the axial displacement

$$u(x) = \delta \exp\left(\frac{2x}{L}\right) - 1\text{mm}, \quad \delta = 0.40\text{ mm}, \quad 0 \leq x \leq L.$$

1. Find the strain  $\varepsilon(x)$  and the total elongation.
2. Would adding a constant displacement of 1 mm everywhere change the strain?

### Solution

Differentiating with respect to the original coordinate gives

$$\varepsilon(x) = \frac{2\delta}{L} \exp\left(\frac{2x}{L}\right).$$

The total elongation is the difference between the end displacements:

$$\Delta L = u(L) - u(0) = \delta(e^2 - 1) \approx 2.56\text{ mm}.$$

Adding a constant displacement does not change its derivative; it is a rigid body translation:

$$\frac{d}{dx}(u(x) + 1\text{ mm}) = \frac{du}{dx}.$$

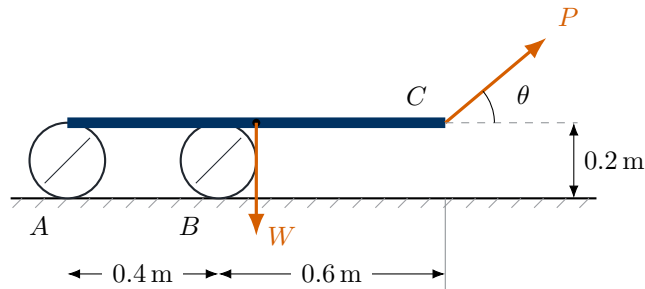
**Example 2: Keeping a cart at rest**

A cart is modeled as a uniform horizontal beam of length 1 m and weight  $W = 200\sqrt{2}$  N. Its rear and front ground contacts,  $A$  and  $B$ , are located 0.4 m apart. Contact  $A$  lies directly below the rear end of the beam.

The weight acts at the beam's center, 0.1 m ahead of contact  $B$ . A force  $P = 100$  N pulls upward and to the right at the front end  $C$ , which is 0.2 m above the ground.

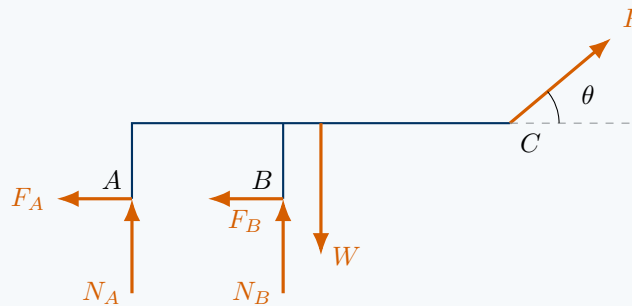
Both sets of wheels are locked and have the same coefficient of static friction  $\mu_s$ .

Determine the smallest angle  $\theta \in [0, \pi/2]$  for which the cart does not tip, assuming sufficient friction. At this angle, determine the minimum value of  $\mu_s$  needed to prevent sliding.



**Solution**

The weight tends to tip the cart clockwise about the front contact  $B$ , lifting the rear contact  $A$ . The upward component of  $P$  opposes this tendency.



At the tipping limit about  $B$ , the rear reaction is  $N_A = 0$ . Taking counterclockwise moments about  $B$  gives

$$-(0.1 \text{ m})W + (0.6 \text{ m})P \sin \theta - (0.2 \text{ m})P \cos \theta = 0.$$

Substituting  $W = 200\sqrt{2} \text{ N}$  and  $P = 100 \text{ N}$ ,

$$3 \sin \theta - \cos \theta = \sqrt{2}.$$

This is satisfied by

$$3 \sin \frac{\pi}{4} - \cos \frac{\pi}{4} = \frac{3-1}{\sqrt{2}} = \sqrt{2}.$$

The expression  $3 \sin \theta - \cos \theta$  increases on  $[0, \pi/2]$ , so smaller angles provide insufficient counterclockwise moment. Therefore,

$$\theta_{\min} = \frac{\pi}{4} = 45^\circ.$$

At this angle, vertical equilibrium gives

$$N_B = W - P \sin \frac{\pi}{4} = 200\sqrt{2} - 50\sqrt{2},$$

so

$$N_A = 0, \quad N_B = 150\sqrt{2} \text{ N}.$$

Since  $N_A = 0$ , the rear contact supplies no friction. Horizontal equilibrium requires

$$F_B = P \cos \frac{\pi}{4} = 50\sqrt{2} \text{ N}$$

to the left. To prevent sliding,

$$F_B \leq \mu_s N_B.$$

Hence,

$$\mu_{s,\min} = \frac{50\sqrt{2}}{150\sqrt{2}} = \frac{1}{3}.$$

At  $\theta = 45^\circ$ , tipping is impending. If also  $\mu_s = 1/3$ , sliding is impending at the same time.

## 4 Challenge exercise

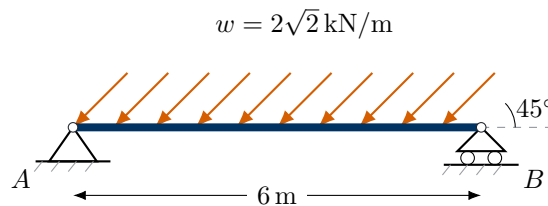
### Example 3: Internal forces under an inclined distributed load

A weightless beam  $AB$  is supported by a pin at  $A$  and a roller on a horizontal surface at  $B$ . The total beam length is 6 m.

A uniform distributed load of magnitude  $w = 2\sqrt{2}$  kN/m acts over the entire beam, directed downward and to the left at  $45^\circ$  to the horizontal. The load intensity is specified per unit length of the beam.

Measure  $x$  from  $A$ , positive to the right.

1. Determine the axial force  $N(x)$  and draw the axial force diagram.
2. Determine the shear force  $V(x)$  and bending moment  $M(x)$  along the beam and draw their diagrams.



### Solution

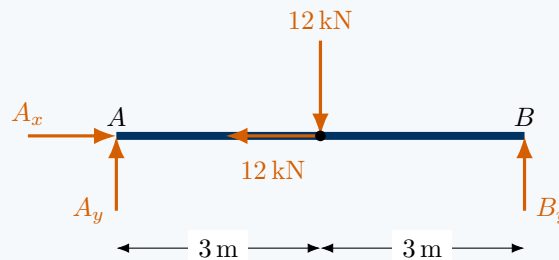
The horizontal and vertical load components have equal magnitudes:

$$q = w \cos 45^\circ = w \sin 45^\circ = 2 \text{ kN/m}.$$

Thus, the load per unit length is

$$\mathbf{f} = -q\mathbf{e}_x - q\mathbf{e}_y.$$

For the whole beam, each component has a resultant magnitude  $qL = 12$  kN, acting at the midpoint.



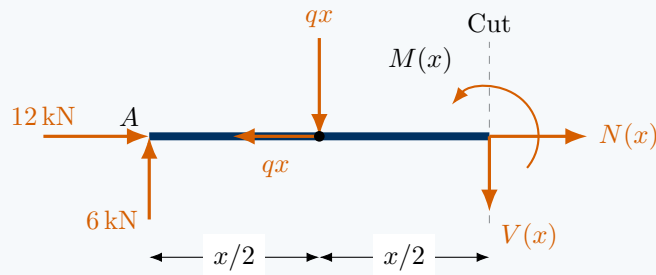
Taking moments about  $A$ ,

$$B_y(6 \text{ m}) - (12 \text{ kN})(3 \text{ m}) = 0.$$

The horizontal load component acts along the beam axis and has no moment about  $A$ . Force equilibrium then gives

$$\boxed{A_x = 12 \text{ kN}, \quad A_y = B_y = 6 \text{ kN}.}$$

Now cut at a position  $x$  and isolate the left portion. Only the load acting over this portion is included. Its horizontal and vertical resultants each have magnitude  $qx$  and act at  $x/2$ .



**1. Axial force.** Horizontal equilibrium gives

$$12 \text{ kN} - qx + N(x) = 0.$$

Therefore,

$$N(x) = (2 \text{ kN/m})x - 12 \text{ kN}, \quad 0 < x < 6 \text{ m}.$$

The axial force is negative throughout the interior of the beam, so the beam is in compression. It varies linearly from  $-12 \text{ kN}$  just to the right of  $A$  to zero at  $B$ .

**2. Shear force and bending moment.** Vertical equilibrium gives

$$6 \text{ kN} - qx - V(x) = 0,$$

so

$$V(x) = 6 \text{ kN} - (2 \text{ kN/m})x, \quad 0 < x < 6 \text{ m}.$$

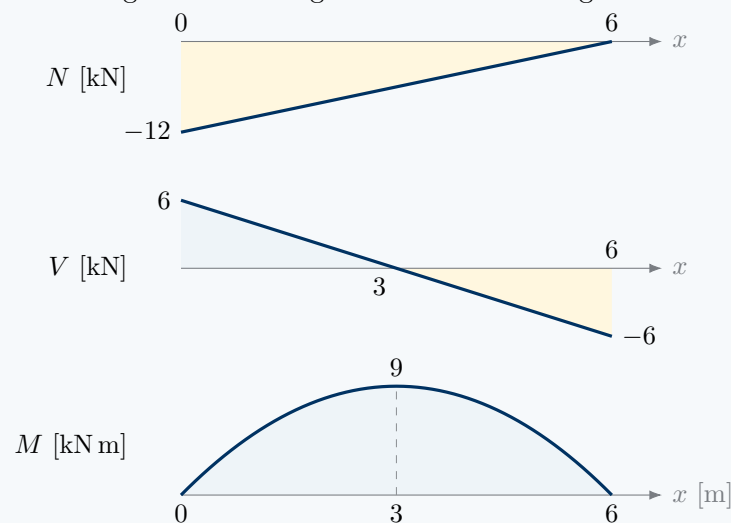
Taking counterclockwise moments about the cut,

$$M(x) - (6 \text{ kN})x + (qx)\frac{x}{2} = 0.$$

Thus,

$$M(x) = (6 \text{ kN})x - (1 \text{ kN/m})x^2, \quad 0 \leq x \leq 6 \text{ m}.$$

The axial and shear force diagrams are straight lines. The bending moment diagram is a parabola.



The bending moment is zero at both supports. Completing the square gives

$$M(x) = 9 \text{ kN m} - (1 \text{ kN/m})(x - 3 \text{ m})^2,$$

so the maximum is

$$M_{\max} = 9 \text{ kN m} \quad \text{at } x = 3 \text{ m}.$$