

Discussion 05

Weekly Material Recap and Example Problems

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1 General reminders

- **Material coverage reminder:** We have moved to mechanics of materials, meaning that the reference text is now [G]: *Engineering Mechanics of Deformable Solids* by Professor Govindjee.
- **Future regrade requests:** Please submit regrade requests as quickly as possible, ideally within 1–2 days of grades being posted. The GSIs had to stop considering requests after Saturday.
- **GSI logistics:** This week and next (October 1 and 8), I will be absent from discussion and Thursday office hours. Someone else will cover both discussion sections, but both Thursday office hours will be canceled.
- **Solution tip:** In an indeterminate problem, it can sometimes be useful to treat one of the unknowns as though it is known, and then use kinematics and the constitutive equations to solve for that unknown. For linear elastic problems, one can also split a problem into multiple statically determinate problems and solve via superposition, but this can increase the likelihood of making a mistake!

2 September 21–September 25: material recap

2.1 One-dimensional mechanics of materials under static conditions

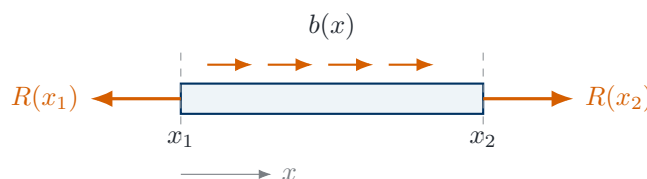
In general, combining kinematics and constitutive assumptions with standard force equilibrium allows us to solve problems that would otherwise be statically indeterminate.

Consider a straight bar with axial displacement $u(x)$. Let $R(x)$ denote the internal axial force, with **positive R indicating tension**. Let $b(x)$ denote the externally applied axial force per unit length, positive in the x -direction.

Isolating a segment between x_1 and x_2 , force equilibrium gives

$$R(x_2) - R(x_1) + \int_{x_1}^{x_2} b(x) dx = \int_{x_1}^{x_2} \left[\frac{dR}{dx} + b(x) \right] dx = 0.$$

The first two terms are the forces acting on the cut faces; the integral is the resultant of the distributed load on the segment.



Positive internal forces pull outward on both cut faces.

Because equilibrium must hold for any arbitrary segment, the local equilibrium equation is

$$\frac{dR}{dx} + b(x) = 0.$$

This connects the applied loading to the variation of internal force along the bar.

One-dimensional mechanics equations

For an axially loaded, linear elastic bar, we use four relations:

- **Force equilibrium:**

$$\frac{dR}{dx} + b(x) = 0,$$

where R is the internal axial force [N] and b is the external axial force per unit length [N/m].

- **Section force resultant:**

$$R(x) = \sigma(x)A(x),$$

where A is the cross-sectional area [m²] and σ is the axial stress [N/m²], treated as uniform over each cross section in one-dimension.

- **Linear elastic constitutive relation:**

$$\sigma(x) = E(x)\epsilon(x),$$

where E is the elastic modulus [N/m²] and ϵ is the axial strain [m/m].

- **Kinematics:**

$$\epsilon(x) = \frac{du}{dx},$$

where u is the axial displacement [m].

Putting everything together gives the ordinary differential equation

$$\boxed{\frac{d}{dx} \left[E(x)A(x) \frac{du}{dx} \right] + b(x) = 0.} \quad (1)$$

To determine the displacement solutions, we also need **two boundary conditions**, usually one at each end of the bar. At each end, we generally prescribe either the displacement u or the axial force; for example, a fixed end has zero displacement, while a free end has zero axial force. In some cases, such as an end attached to a spring, we instead know a relation between force and displacement.

Once $u(x)$ is known, the equations above give the strain, stress, and internal force.

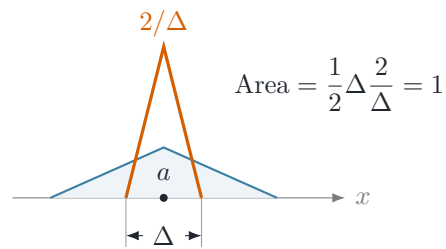
2.2 Singularity functions

Although a true point force does not exist, we have seen time and time again that idealizing a narrowly distributed force as a point force is useful. Analytical tools called **singularity functions** allow us to represent these concentrated loads, along with loads that start or stop abruptly, when integrating (1).

Dirac delta:

The Dirac delta $\delta(x - a)$ represents a unit singularity at $x = a$: it integrates to one over any interval containing a in its interior, and to zero otherwise. For example, a point force P is represented by

$$b(x) = P\delta(x - a), \quad \int_{x_1}^{x_2} b(x) dx = P \quad (x_1 < a < x_2).$$



As $\Delta \rightarrow 0$, the unit-area triangular distribution approaches $\delta(x - a)$.

Heaviside step function:

The Heaviside step function switches from zero to one at $x = a$:

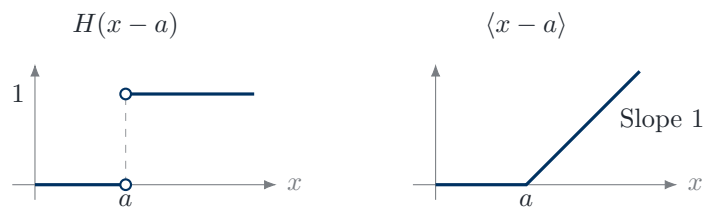
$$H(x - a) = \begin{cases} 0, & x < a, \\ 1, & x > a. \end{cases}$$

The value at a is undefined.

Macauley brackets

The Macauley bracket is zero before a and increases linearly after a :

$$\langle x - a \rangle = \begin{cases} 0, & x < a, \\ x - a, & x \geq a. \end{cases}$$



Integration rules

Integrating the delta gives a step; integrating the step gives a ramp:

$$\int \delta(x - a) dx = H(x - a) + C, \quad \int H(x - a) dx = \langle x - a \rangle + C.$$

For positive integers n ,

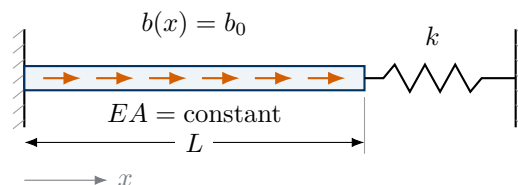
$$\int \langle x - a \rangle^n dx = \frac{\langle x - a \rangle^{n+1}}{n + 1} + C.$$

3 Example problems

Report answers with appropriate units. Draw a free body diagram where applicable.

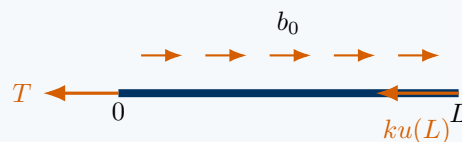
Example 1: A spring-supported bar ([G], Exercise 2.17)

A bar of length L and constant axial rigidity EA is fixed at $x = 0$. Its right end is attached to a linear spring of stiffness k , whose other end is fixed. The spring is initially undeformed. A uniform axial load $b(x) = b_0$ acts along the bar in the positive x -direction. Using the differential equation for axial equilibrium, determine the displacement $u(x)$.



Solution

The left end is fixed, so $u(0) = 0$. The spring opposes the displacement of the right end: its force on the bar is $-ku(L)$.



Thus, the boundary conditions are

$$u(0) = 0, \quad R(L) = EAu'(L) = -ku(L).$$

Since EA is constant, equilibrium gives

$$EAu''(x) + b_0 = 0.$$

Integrating twice,

$$u'(x) = -\frac{b_0}{EA}x + C_0, \quad u(x) = -\frac{b_0x^2}{2EA} + C_0x + C_1.$$

The condition $u(0) = 0$ gives $C_1 = 0$. At $x = L$,

$$u'(L) = -\frac{b_0L}{EA} + C_0 = -\frac{k}{EA} \left(-\frac{b_0L^2}{2EA} + C_0L \right).$$

Solving,

$$C_0 = \frac{b_0L(2EA + kL)}{2EA(EA + kL)}.$$

Therefore,

$$u(x) = \frac{b_0}{2EA} \left[\frac{L(2EA + kL)}{EA + kL} x - x^2 \right].$$

In particular,

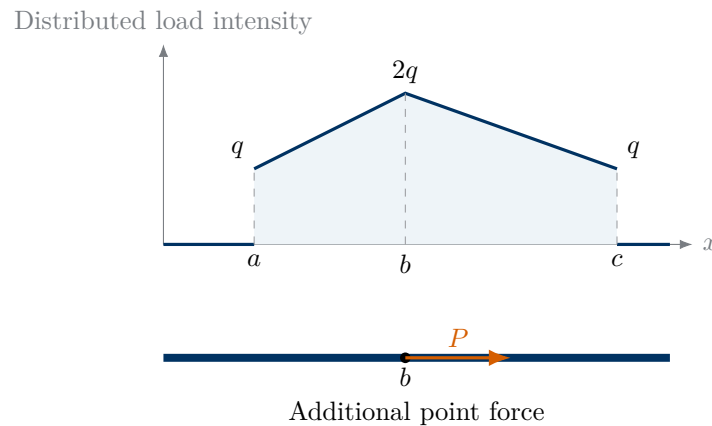
$$u(L) = \frac{b_0L^2}{2(EA + kL)}.$$

Example 2: Representing a load with singularity functions

An axial distributed load acts in the positive x -direction. Its intensity is zero for $x < a$, increases linearly from q at $x = a$ to $2q$ at $x = b$, and then decreases linearly to q at $x = c$. It is zero for $x > c$, where $a < b < c$.

An additional point force P acts in the positive x -direction at $x = b$.

Write a single expression for the total loading $b(x)$ using singularity functions. Here, q has units of force per unit length and P has units of force.

**Solution**

The two slopes of the distributed-load diagram are

$$m_1 = \frac{q}{b-a}, \quad m_2 = -\frac{q}{c-b}.$$

At a , the load jumps up by q and its slope becomes m_1 . At b , the slope changes from m_1 to m_2 . At c , the slope returns to zero and the load drops by q . Thus,

$$\begin{aligned} b(x) = & q[H(x-a) - H(x-c)] \\ & + \frac{q}{b-a}\langle x-a \rangle \\ & - \left(\frac{q}{b-a} + \frac{q}{c-b} \right) \langle x-b \rangle \\ & + \frac{q}{c-b}\langle x-c \rangle + P\delta(x-b). \end{aligned}$$

The step functions account for the jumps in load intensity, the brackets account for changes in slope, and the delta represents the additional point force.

As a check, for $x > c$ the ramp terms reduce to

$$\frac{q}{b-a}[(x-a) - (x-b)] + \frac{q}{c-b}[(x-c) - (x-b)] = q - q = 0.$$

The step terms also cancel, so the distributed load vanishes beyond c .

4 Challenge exercise

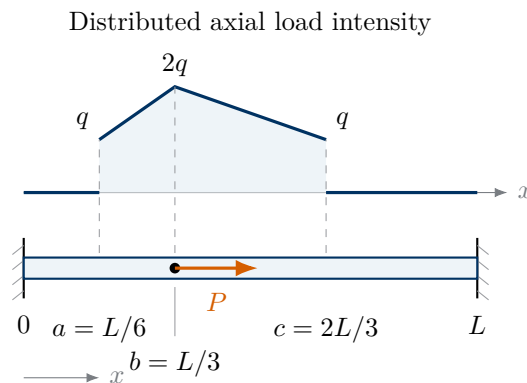
Example 3: A fixed-ended bar with combined loading

A bar of length L and constant axial rigidity EA is fixed at both ends. It carries the distributed load and point force from the previous problem, with

$$a = \frac{L}{6}, \quad b = \frac{L}{3}, \quad c = \frac{2L}{3}.$$

All applied loads act in the positive x -direction.

Determine the displacement $u(x)$, internal axial force $R(x)$, and reactions at both ends.



Solution

The boundary conditions are

$$u(0) = u(L) = 0.$$

The ascending and descending slopes of the load diagram are $6q/L$ and $-3q/L$, respectively. Thus,

$$b(x) = q[H(x-a) - H(x-c)] + \frac{6q}{L}\langle x-a \rangle - \frac{9q}{L}\langle x-b \rangle + \frac{3q}{L}\langle x-c \rangle + P\delta(x-b).$$

Integrating $R'(x) = -b(x)$ gives

$$R(x) = C_1 - q[\langle x-a \rangle - \langle x-c \rangle] - \frac{3q}{L}\langle x-a \rangle^2 + \frac{9q}{2L}\langle x-b \rangle^2 - \frac{3q}{2L}\langle x-c \rangle^2 - PH(x-b).$$

Since $EAu'(x) = R(x)$, integrating again gives

$$EAu(x) = C_1x + C_2 - \frac{q}{2}[\langle x-a \rangle^2 - \langle x-c \rangle^2] - \frac{q}{L}\langle x-a \rangle^3 + \frac{3q}{2L}\langle x-b \rangle^3 - \frac{q}{2L}\langle x-c \rangle^3 - P\langle x-b \rangle.$$

The condition $u(0) = 0$ gives $C_2 = 0$. Using $u(L) = 0$,

$$\begin{aligned} 0 &= C_1 L - \frac{q}{2} \left[\left(\frac{5L}{6} \right)^2 - \left(\frac{L}{3} \right)^2 \right] \\ &\quad - \frac{q}{L} \left(\frac{5L}{6} \right)^3 + \frac{3q}{2L} \left(\frac{2L}{3} \right)^3 \\ &\quad - \frac{q}{2L} \left(\frac{L}{3} \right)^3 - \frac{2PL}{3} \\ &= C_1 L - \frac{4}{9} q L^2 - \frac{2}{3} PL. \end{aligned}$$

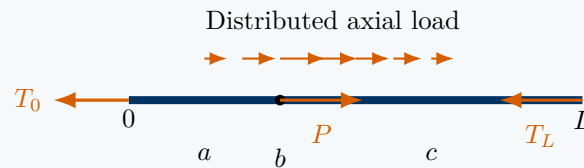
Therefore,

$$C_1 = \frac{4}{9} q L + \frac{2}{3} P, \quad C_2 = 0.$$

Substituting these constants into the boxed expressions determines $R(x)$ and $u(x)$.

Support reactions.

Let T_0 and T_L denote the magnitudes of the leftward reactions.



The left reaction has magnitude $T_0 = R(0) = C_1$. The resultant of the distributed load is

$$Q = \frac{3q}{2}(c-a) = \frac{3}{4}qL.$$

Whole-bar equilibrium gives

$$T_0 + T_L = Q + P.$$

Hence,

$$T_0 = \frac{4}{9} q L + \frac{2}{3} P, \quad T_L = \frac{11}{36} q L + \frac{1}{3} P$$

with both reactions acting to the left.