CE 30 / ME 85 — Introduction to Solid Mechanics **Fall 2026**

Discussion 07

Weekly Material Recap and Example Problems

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1 General reminders

- **Material coverage reminder:** We have now covered through the beginning of Chapter 5 in Professor Govindjee's book, including three-dimensional strain and linear elastic constitutive relations. This week, he will cover plane problems.
- **GSI logistics:** I am finally back from my travels and will be here this week and next. Another GSI will cover discussion on October 29 while I attend ECS 250. My office hours will be canceled that week.
- **Midterm 2 reminder:** Material coverage begins with axial bar problems and will end with torsion, which is coming up.

2 October 5–October 9: material recap

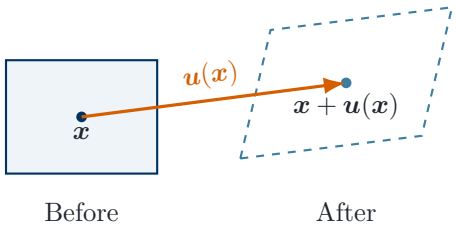
2.1 Full three-dimensional displacements

In one dimension, we described deformation using a single displacement $u(x)$. In three dimensions, each point can move in three directions:

$$\boldsymbol{u}(x,y,z) = u(x,y,z)\boldsymbol{e}_x + v(x,y,z)\boldsymbol{e}_y + w(x,y,z)\boldsymbol{e}_z.$$

Here, (x,y,z) identifies the point *before* deformation. Its new position is

$$\boxed{\boldsymbol{x}_{\text{new}} = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})}.$$



A two-dimensional sketch of a displacement field.

As before, deformation depends on **infinitesimal differences between displacements**.

2.2 Full three-dimensional strain definition

Normal strains describe changes in length; shear strains describe changes in the angles between initially perpendicular material lines. For small displacement gradients, both are described by the **infinitesimal strain tensor**.

Three-dimensional strain

The infinitesimal strain tensor is symmetric, and so

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{pmatrix}.$$

The normal strains are

$$\epsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}, \qquad \epsilon_{zz} = \frac{\partial w}{\partial z}.$$

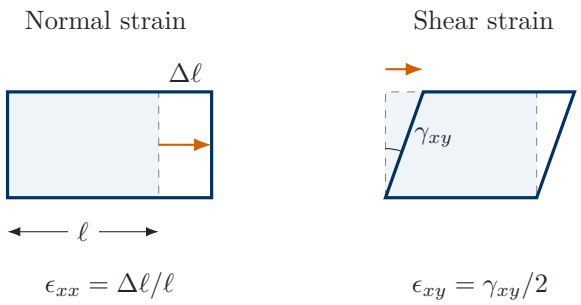
The shear strains are

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$
$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$
$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).$$

The **engineering shear strain** is twice the corresponding tensor component:

$$\gamma_{xy} = 2\epsilon_{xy}, \qquad \gamma_{xz} = 2\epsilon_{xz}, \qquad \gamma_{yz} = 2\epsilon_{yz}.$$

For small deformation, γ_{xy} is the decrease in the initially right angle between material lines along x and y, measured in radians.



Dashed outlines show the original shape. Deformations are exaggerated.

Positive normal strain indicates extension; negative normal strain indicates shortening.

2.3 Three-dimensional linear elastic constitutive equations

Constitutive equations connect stress to strain. For an **isotropic** material, the elastic response is the same in every direction. Two independent elastic constants are sufficient to fully characterize response: Young's modulus E and Poisson's ratio ν.

Unlike the one-dimensional relation, a normal strain generally depends on all three normal stresses.

Isotropic linear elasticity

In the absence of thermal strain,

$$\epsilon_{xx} = \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})],$$
$$\epsilon_{yy} = \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})],$$
$$\epsilon_{zz} = \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})].$$

The shear relations are

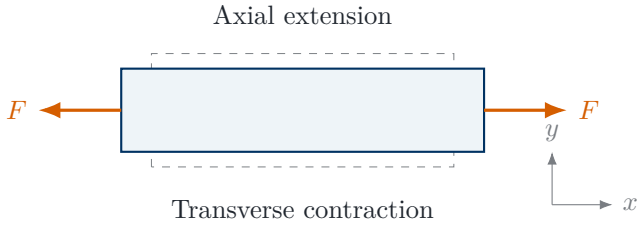
$$\sigma_{xy} = 2G\epsilon_{xy} = G\gamma_{xy}, \qquad \sigma_{xz} = 2G\epsilon_{xz} = G\gamma_{xz},$$
$$\sigma_{yz} = 2G\epsilon_{yz} = G\gamma_{yz}, \qquad \boxed{G = \frac{E}{2(1+\nu)}}.$$

Here, G is the shear modulus. Both E and G have units of stress; ν is dimensionless.

For example, under uniaxial stress with only σ_{xx} nonzero,

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E}, \qquad \epsilon_{yy} = \epsilon_{zz} = -\nu\frac{\sigma_{xx}}{E}.$$

Thus, for $\nu > 0$, stretching a bar makes it contract in the transverse directions.



Dashed outline: before loading. Solid outline: after loading, exaggerated.

Observe that zero stress in a direction does not imply zero strain in that direction.

3 Example problems

Example 1: Interpreting strain in cylindrical coordinates

In cylindrical coordinates (r, θ, z), the infinitesimal strain tensor is given below. Its rows and columns correspond to the r, θ, and z directions, respectively.

$$\boldsymbol{\epsilon} = \begin{pmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r}\right) & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta}\right) & \dfrac{\partial u_z}{\partial z} \end{pmatrix}.$$

Explain the following in your own words, using sketches where helpful.

1. What changes in length or angle do the diagonal and off-diagonal components describe?
2. In $\epsilon_{\theta\theta}$, why is the angular derivative divided by r? Why does radial displacement contribute the term u_r/r?
3. Why is the term $-u_\theta/r$ needed in $\epsilon_{r\theta}$? Consider a small rigid rotation:

$$u_r = u_z = 0, \qquad u_\theta = \alpha r, \qquad |\alpha| \ll 1.$$

Solution

1. Lengths and angles.
Imagine a small material element with edges along the local r, θ, and z directions. The diagonal entries give the fractional changes in its radial length, circumferential arc length, and axial length, respectively.
Each off-diagonal entry is half the decrease in the initially right angle between its two material directions. For example, $2\epsilon_{r\theta}$ is the decrease in the angle between the radial and circumferential edges.

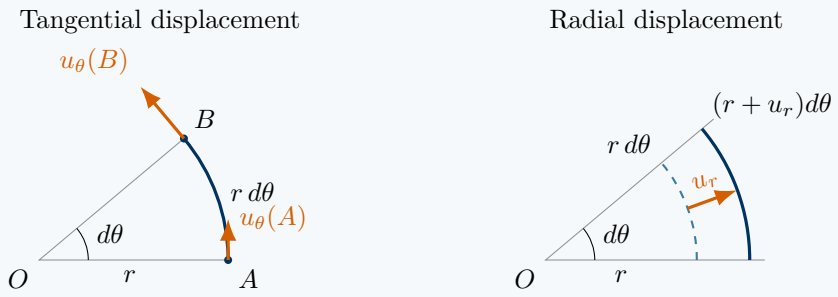
2. Two ways to stretch a circumferential arc.
Different tangential displacements. Two nearby points A and B at the same radius are initially separated by $r\,d\theta$. If B moves farther around the circle than A, their separation increases. Dividing this displacement difference by the original arc length gives

$$\frac{u_\theta(\theta + d\theta) - u_\theta(\theta)}{r\,d\theta} \longrightarrow \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}.$$

A larger radius. If both points move radially outward by u_r, keeping the same angular separation, the arc also gets longer:

$$\frac{(r + u_r)d\theta - r\,d\theta}{r\,d\theta} = \frac{u_r}{r}.$$

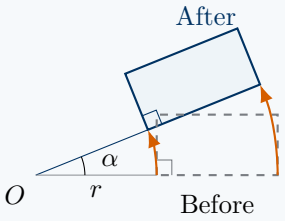
These two contributions add to give $\epsilon_{\theta\theta}$.



View along the z-axis; displacements are exaggerated. On the right, the dashed arc is the original position.

3. Rotation without shear.

During a small rigid rotation through α, a point at radius r moves tangentially by $u_\theta = \alpha r$. More distant points move farther, but both edges of a material element rotate by the same angle. Their right angle is unchanged.



Both edges rotate.
The right angle stays 90°.

The derivative $\partial u_\theta/\partial r = \alpha$ alone would count this rotation as shear. The term $-u_\theta/r = -\alpha$ cancels it:

$$\epsilon_{r\theta} = \frac{1}{2}\,(0 + \alpha - \alpha) = 0.$$

Example 2: Strain from a displacement field

A body occupying $0 \le x, y, z \le L$ has displacement components

$$u(x,y,z) = \frac{\alpha}{L}(x^2 + 2yz),$$

$$v(x,y,z) = \frac{\alpha}{L}(y^2 - 2xz), \qquad w(x,y,z) = \frac{\alpha}{L}z^2,$$

where L is a length and $\alpha = 10^{-3}$ is dimensionless.
Determine the infinitesimal strain tensor $\boldsymbol{\epsilon}(x,y,z)$ throughout the body.

Solution

The normal strains are

$$\epsilon_{xx} = \frac{2\alpha x}{L}, \qquad \epsilon_{yy} = \frac{2\alpha y}{L}, \qquad \epsilon_{zz} = \frac{2\alpha z}{L}.$$

For the shear components,

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{2\alpha z}{L} - \frac{2\alpha z}{L}\right) = 0,$$

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{2\alpha y}{L} + 0\right) = \frac{\alpha y}{L},$$

$$\epsilon_{yz} = \frac{1}{2}\left(-\frac{2\alpha x}{L} + 0\right) = -\frac{\alpha x}{L}.$$

Therefore,

$$\boxed{\boldsymbol{\epsilon}(x,y,z) = \frac{\alpha}{L}\begin{pmatrix} 2x & 0 & y \\ 0 & 2y & -x \\ y & -x & 2z \end{pmatrix}.}$$

4 Challenge exercise

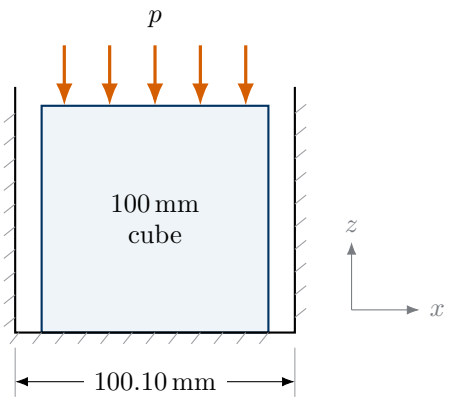
Example 3: Compression of a block with clearance

An initially unstressed, isotropic linear elastic cube has side length 100 mm and material properties

$$E = 2\,\text{GPa}, \qquad \nu = 0.25.$$

It rests on a rigid base and is centered between four rigid sidewalls forming a square opening of width 100.10 mm in both the x and y directions. Thus, the initial gap at each sidewall is 0.05 mm. A uniform downward pressure p is gradually applied to the top face. Ignore weight and friction, and assume uniform stress and strain in the block.

1. Determine the pressure at which the block first touches the sidewalls. How much has its height decreased at this point?
2. Determine the pressure needed to decrease the original height by 0.80 mm, and the pressure exerted by each sidewall.



Section in the xz-plane. The same clearance exists in the y-direction. Gaps are exaggerated.

Solution

1. Before contact with the sidewalls.
The lateral faces are traction-free, so

$$\sigma_{xx} = \sigma_{yy} = 0, \qquad \sigma_{zz} = -p.$$

The constitutive relations give

$$\epsilon_{xx} = \epsilon_{yy} = \frac{\nu p}{E}, \qquad \epsilon_{zz} = -\frac{p}{E}.$$

The block touches the walls when its width increases by 0.10 mm:

$$\epsilon_{xx} = \epsilon_{yy} = \frac{0.10}{100} = 0.001.$$

Thus,

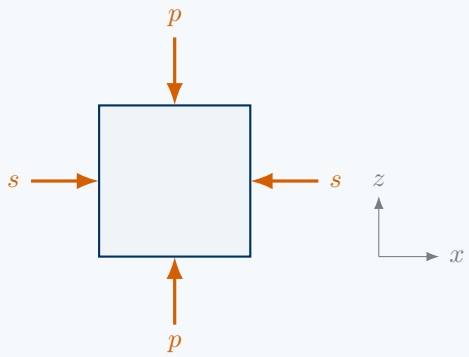
$$\boxed{p_c = \frac{E(0.001)}{\nu} = 8\,\text{MPa}.}$$

The decrease in height is

$$\boxed{\Delta h_c = \frac{p_c}{E}(100\,\text{mm}) = 0.40\,\text{mm}.}$$

2. After contact with the sidewalls.
Since $0.80\,\text{mm} > 0.40\,\text{mm}$, the block is in contact with the walls. Let s denote the compressive pressure exerted by each sidewall:

$$\sigma_{xx} = \sigma_{yy} = -s, \qquad \sigma_{zz} = -p.$$



Compressive normal tractions after contact. The two y-faces also carry pressure s.

The walls prevent *further* lateral expansion. The total lateral strains remain 0.001:

$$\epsilon_{xx} = \epsilon_{yy} = 0.001, \qquad \epsilon_{zz} = -\frac{0.80}{100} = -0.008.$$

Applying the constitutive relations,

$$E(0.001) = -(1-\nu)s + \nu p,$$
$$E(-0.008) = -p + 2\nu s.$$

With $E = 2000\,\text{MPa}$ and $\nu = 0.25$,

$$2\,\text{MPa} = -0.75s + 0.25p, \qquad -16\,\text{MPa} = -p + 0.50s.$$

Solving,

$$\boxed{p = 17.6\,\text{MPa}}, \qquad \boxed{s = 3.2\,\text{MPa}}.$$