CE 30 / ME 85 — Introduction to Solid Mechanics Fall 2026

Discussion 08

Weekly Material Recap and Example Problems

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1 General reminders

- **Assignment reminder:** Homework 8 is due Friday, October 23, at 5:59 pm.
- **Midterm 2:** The exam will be Monday, November 2. Material coverage begins with axial bar problems and ends with torsion.
- **GSI logistics:** Another GSI will cover discussion on October 29 while I attend ECS 250. My office hours will be canceled that week.

2 October 12–October 16: material recap

2.1 Constitutive relations and thermal strain

We can include a change in temperature in the constitutive equations by adding the strain that would occur if the material were free to expand.

Isotropic thermoelasticity

For a temperature change ΔT from an initially unstressed reference state,

$$\epsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} + \alpha \Delta T,$$

$$\sigma_{xy} = 2G\epsilon_{xy} = G\gamma_{xy}, \qquad G = \frac{E}{2(1+\nu)}.$$

The other components follow the same pattern. The thermal expansion coefficient α has units of inverse temperature; the shear relations are unchanged.

Unrestricted heating produces strain without stress. Preventing expansion causes stress to develop. Below, we take $\Delta T = 0$ unless stated otherwise.

2.2 Pressure and changes in volume

For uniform pressure, we take $p > 0$ in compression: $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, with zero shear stresses.

Bulk modulus

For small strains, the fractional change in volume is

$$\frac{\Delta V}{V} \approx \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = -\frac{3(1-2\nu)p}{E} = -\frac{p}{K}, \qquad K = \frac{E}{3(1-2\nu)}.$$

The bulk modulus K measures resistance to a change in volume.

2.3 Plane stress, plane strain, and uniaxial stress

These are useful special cases of the three-dimensional equations. The assumptions concern either stress or deformation, so it is important to identify which quantity is constrained.

Plane stress

A thin plate loaded in its plane, with traction-free faces normal to z, can often be approximated by

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0.$$

The remaining relations are

$$\epsilon_{xx} = \frac{\sigma_{xx} - \nu\sigma_{yy}}{E}, \quad \epsilon_{yy} = \frac{\sigma_{yy} - \nu\sigma_{xx}}{E},$$
$$\gamma_{xy} = \frac{\sigma_{xy}}{G}, \qquad \epsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}).$$

Observe that the thickness can change even though $\sigma_{zz} = 0$.

Plane strain

When extension in z is prevented and the response does not vary along z, we use

$$\epsilon_{zz} = \epsilon_{xz} = \epsilon_{yz} = 0.$$

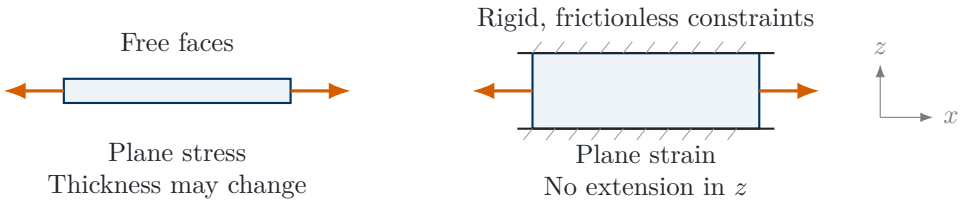
Setting $\epsilon_{zz} = 0$ in the constitutive equation gives

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}).$$

Substitution into the other equations gives

$$\epsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right],$$
$$\epsilon_{yy} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right], \qquad \gamma_{xy} = \frac{\sigma_{xy}}{G}.$$

A nonzero σ_{zz} is generally needed to prevent extension in z.



Side views in the xz-plane; loading is along x.

For **uniaxial stress**, only σ_{xx} is nonzero:

$$\epsilon_{xx} = \frac{\sigma_{xx}}{E}, \qquad \epsilon_{yy} = \epsilon_{zz} = -\nu\epsilon_{xx}.$$

Our one-dimensional bar model determines the axial response. The lateral contraction can then be recovered from these equations.

2.4 Axial loading of a composite section

For uniform axial extension of a composite bar, the bonded materials have the same axial strain. Each develops stress according to its own constitutive law.

Compatibility, constitutive response, and equilibrium

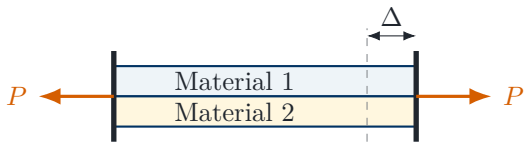
For two elastic materials in a uniform bar of length L,

$$\epsilon_1 = \epsilon_2 = \frac{\Delta}{L}, \qquad \sigma_i = E_i \frac{\Delta}{L}, \quad i = 1, 2.$$

The internal axial force is the sum of the forces in the materials:

$$R = (E_1 A_1 + E_2 A_2)\frac{\Delta}{L}, \qquad \sigma_i = \frac{E_i R}{E_1 A_1 + E_2 A_2}.$$

More generally, $R = \int_A \sigma_{xx}\, dA$.



Both materials have the same elongation. The dashed line marks the original right end.

2.5 Thin-walled pressure vessels

With wall thickness $t \ll r$, the in-plane stresses are approximately constant through the wall. Let p be internal gauge pressure. Away from the ends, force equilibrium gives the following results.

Membrane stresses

For a **cylindrical vessel with closed ends**,

$$\boxed{\sigma_{\theta\theta} = \frac{pr}{t}}, \qquad \boxed{\sigma_{zz} = \frac{pr}{2t}}.$$

The hoop stress acts around the circumference; the axial stress acts along the cylinder. For a **spherical vessel**, the stress in either tangential direction is

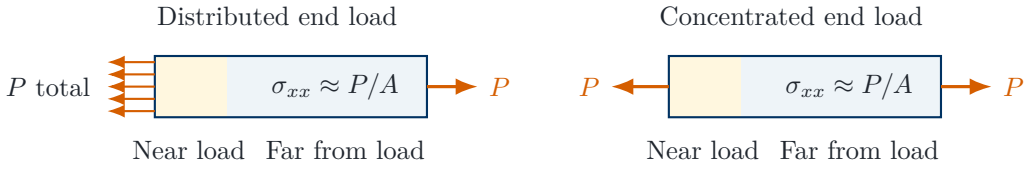
$$\sigma_{\text{tangential}} = \frac{pr}{2t}.$$

Here, r is the mean radius. Differences between inner and mean radius are neglected in this thin-wall approximation.

The radial stress varies from $-p$ at the inner surface to zero at the outer surface, taking exterior pressure as zero. It is small compared with the membrane stresses when t/r is small, so the wall is approximately in plane stress.

2.6 Saint-Venant's principle

Load distributions with the same resultant force and moment give approximately the same stresses at distances large compared with the size of the loaded region. Near the applied loads, the stresses can be quite different.



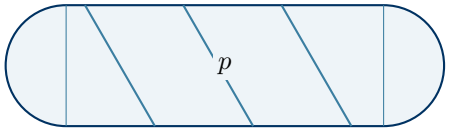
Both end loads have resultant P and zero moment about the section center. The bars have cross-sectional area A.

3 Example problems

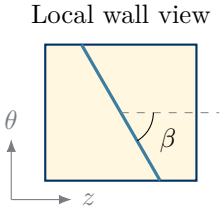
Example 1: A spiral-welded pressure tank

A cylindrical tank has mean radius $r = 500\,\text{mm}$, wall thickness $t = 10\,\text{mm}$, and closed hemispherical end caps. The internal gauge pressure is $p = 2\,\text{MPa}$.
The tank is made from a steel strip welded along a helical seam. The seam makes an acute angle $\beta = 60°$ with the tank axis. Consider the cylindrical wall far from the end caps.

1. Use free body diagrams to determine the hoop stress $\sigma_{\theta\theta}$ and axial stress σ_{zz}.
2. Find the traction on the weld. Resolve it into the normal stress opening the seam and the shear stress along it.



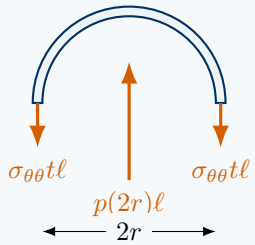
Closed tank; seams shown schematically



Solution

1. Hoop and axial stresses.
Treat the wall stresses as constant through the thickness. Isolate a half-cylinder of length ℓ, then an end cap and its attached wall:

Longitudinal cut



Transverse cut



Pressure resultants shown

The pressure resultants are pressure times projected area. Thus,

$$2\sigma_{\theta\theta} t\ell = p(2r)\ell, \qquad \sigma_{zz}(2\pi rt) = p\pi r^2,$$

so

$$\boxed{\sigma_{\theta\theta} = 100\,\text{MPa}}, \qquad \boxed{\sigma_{zz} = 50\,\text{MPa}}.$$

2. Traction on the weld.
For the seam orientation shown, choose the unit normal $\boldsymbol{n}$ and a unit tangent $\boldsymbol{m}$:

$$\boldsymbol{n} = \sin\beta\,\boldsymbol{e}_z + \cos\beta\,\boldsymbol{e}_\theta, \qquad \boldsymbol{m} = -\cos\beta\,\boldsymbol{e}_z + \sin\beta\,\boldsymbol{e}_\theta.$$

Cauchy's relation gives

$$\boldsymbol{t} = \boldsymbol{\sigma}^T\boldsymbol{n} = \sigma_{zz}\sin\beta\,\boldsymbol{e}_z + \sigma_{\theta\theta}\cos\beta\,\boldsymbol{e}_\theta.$$

At $\beta = 60°$,

$$\boxed{\boldsymbol{t} = (25\sqrt{3}\,\boldsymbol{e}_z + 50\,\boldsymbol{e}_\theta)\,\text{MPa}}.$$

The components relative to the seam are

$$\sigma_n = \boldsymbol{t}\cdot\boldsymbol{n} = \sigma_{zz}\sin^2\beta + \sigma_{\theta\theta}\cos^2\beta = \boxed{62.5\,\text{MPa}},$$

$$\tau = \boldsymbol{t}\cdot\boldsymbol{m} = (\sigma_{\theta\theta} - \sigma_{zz})\sin\beta\cos\beta = \boxed{21.7\,\text{MPa}}.$$

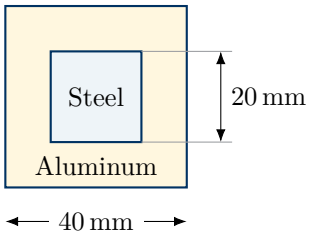
Example 2: A composite bar before and after yielding

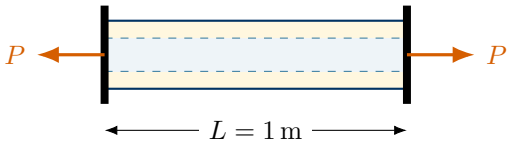
A bar of length $L = 1\,\text{m}$ has a $40\,\text{mm} \times 40\,\text{mm}$ square cross-section, with a $20\,\text{mm} \times 20\,\text{mm}$ steel core bonded to an aluminum casing. Rigid end plates apply an axial tensile force P. Use the one-dimensional bar model, with

$$E_s = 200\,\text{GPa}, \qquad E_a = 70\,\text{GPa}.$$

The tensile force is increased gradually from zero. Model the steel as **elastic–perfectly plastic**: it yields at 250 MPa, then continues to deform at that stress. Take the aluminum to remain elastic over the loads considered.

1. Determine the force P_Y at which the steel first yields.
2. The force is then increased to $P = 268\,\text{kN}$. Determine the stress in each material.





Solution

1. First yield.
Both materials have the same axial strain $\epsilon = \Delta/L$. Isolating the right portion gives $R = P$:



With $A_s = 400\,\text{mm}^2$ and $A_a = 1200\,\text{mm}^2$,

$$P = \sigma_s A_s + \sigma_a A_a.$$

At first yield, both materials are still elastic, so

$$\epsilon_Y = \frac{250\,\text{MPa}}{200\,\text{GPa}} = 0.00125, \qquad \sigma_{a,Y} = E_a \epsilon_Y = 87.5\,\text{MPa}.$$

Therefore,

$$P_Y = (250\,\text{MPa})(400\,\text{mm}^2) + (87.5\,\text{MPa})(1200\,\text{mm}^2) = \boxed{205\,\text{kN}}.$$

2. Beyond first yield.
Since $268\,\text{kN} > P_Y$, the steel has yielded and its stress remains $\boxed{\sigma_s = 250\,\text{MPa}}$. Equilibrium gives

$$\sigma_a = \frac{P - \sigma_s A_s}{A_a} = \frac{268\,\text{kN} - 100\,\text{kN}}{1200\,\text{mm}^2} = \boxed{140\,\text{MPa}}.$$

Both materials still have the same total strain, but the steel now also deforms plastically. The aluminum carries all additional load after first yield.

4 Challenge exercise

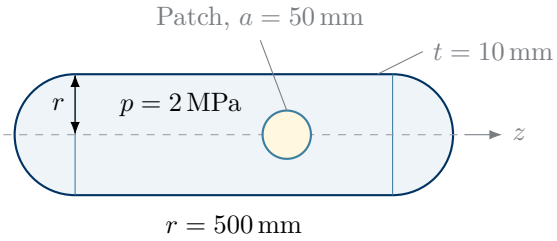
Example 3: Traction around a curved patch

A circular patch of radius $a = 50\,\text{mm}$ is fitted to the cylindrical tank in Example 1, far from the end caps. It has the same material, thickness, and curvature as the wall. Assume it is perfectly joined along its entire edge, so we can use the stresses found in Example 1.
The parameter ϕ is measured at the patch center from the axial direction toward the hoop direction, as shown below. The edge is parametrized by

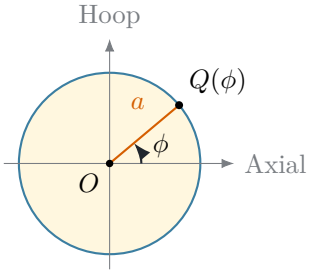
$$z(\phi) = a\cos\phi, \qquad \theta(\phi) = \frac{a}{r}\sin\phi, \qquad 0 \le \phi < 2\pi.$$

Here, z and θ measure axial position and angle about the tank axis, with $z = \theta = 0$ at the patch center.
Along the edge, $\boldsymbol{n}$ is the unit vector perpendicular to the edge, tangent to the wall, and pointing away from the patch.

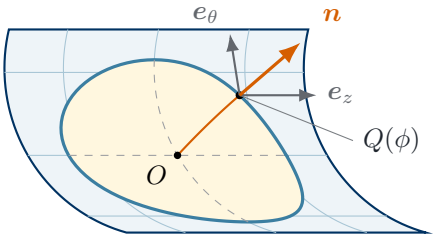
1. Determine the traction $\boldsymbol{t}(\phi)$ exerted on the patch by the surrounding wall. Where is its magnitude largest?
2. Now suppose the patch is attached with bolts. How would this change the stress field near the connection? What else would you need to consider to decide whether it will fail?



Full vessel; patch shown enlarged.



Parameter circle: angle measured at the patch center.



Corresponding point on the curved wall. Curvature exaggerated.

Solution

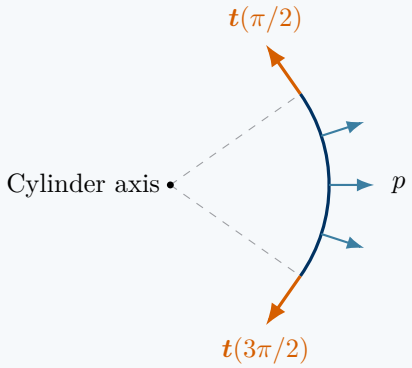
1. Traction around the connection.
The circular parametrization gives the outward unit normal

$$\boldsymbol{n} = \cos\phi\, \boldsymbol{e}_z + \sin\phi\, \boldsymbol{e}_\theta.$$

Using the membrane stresses from Example 1,

$$\boxed{\boldsymbol{t}(\phi) = \boldsymbol{\sigma}^T \boldsymbol{n} = (50\cos\phi\, \boldsymbol{e}_z + 100\sin\phi\, \boldsymbol{e}_\theta)\,\text{MPa}.}$$

The hoop unit vector $\boldsymbol{e}_\theta$ follows the curvature of the cylinder and changes direction along the connection.



Section through the patch center: pressure and edge-traction directions. Curvature exaggerated.

Because $\boldsymbol{e}_z$ and $\boldsymbol{e}_\theta$ are perpendicular unit vectors,

$$|\boldsymbol{t}| = \sqrt{50^2\cos^2\phi + 100^2\sin^2\phi}\,\text{MPa}, \qquad \boxed{|\boldsymbol{t}|_{\max} = 100\,\text{MPa}}$$

at $\phi = \pi/2$ and $3\pi/2$.

2. Replacing the continuous connection with bolts.
Bolts transfer load through discrete contact regions. The holes remove material and create stress concentrations, while differences in stiffness between the bolts, patch, and wall affect load sharing. The smooth traction above therefore does not give the peak stresses in a bolted connection.
Failure could occur in a bolt or in the surrounding wall, for example by bolt shear, bearing at a hole, or tearing between holes. We would need the joint geometry, contact conditions, and material properties to determine the local stresses and compare them with appropriate failure criteria. The nominal membrane stress alone is insufficient.