CE 30 / ME 85 — Introduction to Solid Mechanics **Fall 2026**

Discussion 09

Weekly Material Recap and Example Problems

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Adapted from Ailin Chen's Discussion 9 handout. Thanks Ailin!

1 General reminders

- **Assignment reminder:** Homework 9 is due Friday, October 30. There are nine problems, and this material is included on Midterm 2.
- **Midterm 2:** The exam is Monday, November 2, at the usual lecture time and location. Coverage begins with axial bar problems and ends with torsion. Questions may be multiple choice.

2 October 19–October 23: material recap

2.1 Torsion of circular bars

A torque twists a bar about its axis. Let z measure position along the bar, $\phi(z)$ be the rotation of a cross-section, and $T(z)$ be the internal torque. We again combine equilibrium, kinematics, and a constitutive relation.

Circular bars in torsion

For a circular bar made from a homogeneous, linear elastic material,

$$\text{Equilibrium:} \quad \frac{dT}{dz} + t(z) = 0, \quad T = \int_A \rho\tau \, dA,$$

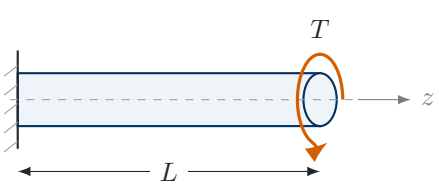
$$\text{Kinematics and elasticity:} \quad \gamma_{z\theta} = \rho\frac{d\phi}{dz}, \qquad \tau = G\gamma_{z\theta}.$$

Here, $t(z)$ is torque per unit length, ρ is distance from the axis, and G is the shear modulus. Combining these relations gives

$$\boxed{T = GJ\frac{d\phi}{dz}}, \qquad \boxed{\tau = \frac{T\rho}{J}}, \qquad J = \int_A \rho^2 \, dA.$$

For a solid circle or a circular annulus,

$$J_{\text{solid}} = \frac{\pi R^4}{2}, \qquad J_{\text{annulus}} = \frac{\pi}{2}(R_o^4 - R_i^4).$$





Cross-section

The governing equation is

$$\boxed{\frac{d}{dz}\left(GJ\frac{d\phi}{dz}\right) + t(z) = 0.}$$

A fixed end prescribes ϕ; an applied end torque prescribes T. For constant T and GJ, the relative rotation over a length L is

$$\Delta\phi = \frac{TL}{GJ}.$$

The torsional rigidity GJ plays the role of EA in an axial-bar problem. The stiffness $T/\Delta\phi$ is GJ/L.

2.2 Thin-walled closed tubes

For a thin-walled tube with one closed cell, the shear flow q is constant around the cross-section. Let $t_w(s)$ be the wall thickness and A_e the area enclosed by the wall's midline.

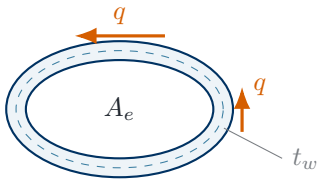
Shear flow and torsional rigidity

$$q = \tau(s)t_w(s) = \frac{T}{2A_e}, \qquad \tau(s) = \frac{T}{2A_e t_w(s)}.$$

For a uniform shear modulus G,

$$\boxed{(GJ)_{\text{eff}} = \frac{4A_e^2 G}{\displaystyle\oint \frac{ds}{t_w(s)}}}, \qquad \frac{d\phi}{dz} = \frac{T}{(GJ)_{\text{eff}}}.$$

The integral follows the wall's midline. For constant torque and cross-section, $\Delta\phi = TL/(GJ)_{\text{eff}}$.



The same shear flow acts around the cell, so thinner portions of the wall carry larger shear stress.

3 Example problems

Report answers with appropriate units. Express angles in radians when using the torsion equations.

Example 1: Power transmitted by a shaft

A steel shaft of diameter 12 mm and length 300 mm rotates at 1000 rpm. The measured relative twist between its ends is 5°. Take $G = 80\,\text{GPa}$.
How much power does the shaft transmit?

Solution

The polar moment and relative twist are

$$J = \frac{\pi(0.012\,\text{m})^4}{32} = 2.036 \times 10^{-9}\,\text{m}^4, \qquad \Delta\phi = 5\frac{\pi}{180} = \frac{\pi}{36}.$$

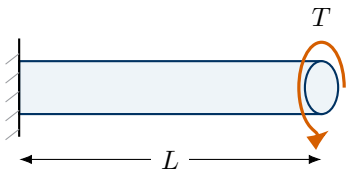
$$T = \frac{GJ\Delta\phi}{L} = 47.4\,\text{N m}.$$

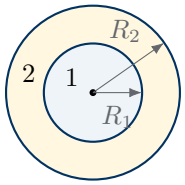
Power is torque times angular speed:

$$\omega = 1000\frac{2\pi}{60} = 104.7\,\text{rad/s}, \qquad \boxed{\mathcal{P} = T\omega \approx 4.96\,\text{kW}.}$$

Example 2: Torsional stiffness of a composite bar

A circular bar of length L consists of a solid core of radius R_1 and shear modulus G_1, surrounded by a jacket with outer radius R_2 and shear modulus G_2. The materials are perfectly bonded and linear elastic. One end is fixed, and a torque T acts at the other end.
Determine the torsional stiffness $T/\Delta\phi$ of the bar.





Cross-section

Solution

Bonding makes the two materials rotate together, so they have the same twist rate $\Delta\phi/L$. Their torques add:

$$T = T_1 + T_2 = G_1J_1\frac{\Delta\phi}{L} + G_2J_2\frac{\Delta\phi}{L},$$

where

$$J_1 = \frac{\pi R_1^4}{2}, \qquad J_2 = \frac{\pi}{2}(R_2^4 - R_1^4).$$

Therefore,

$$\boxed{(GJ)_{\text{eff}} = G_1J_1 + G_2J_2}, \qquad \boxed{\frac{T}{\Delta\phi} = \frac{\pi}{2L}\left[G_1R_1^4 + G_2(R_2^4 - R_1^4)\right].}$$

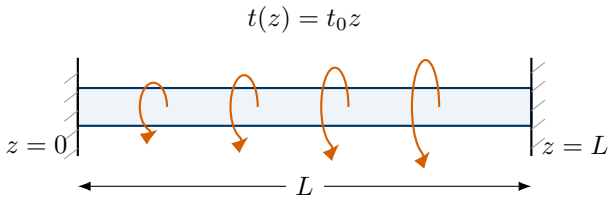
The materials share the twist, but generally carry different torques.

Example 3: First yield in a bar fixed at both ends

A solid circular bar of length L and radius R is fixed at both ends. A distributed torque

$$t(z) = t_0 z, \qquad 0 \le z \le L,$$

acts about its axis, where $t_0 > 0$. The torsional rigidity GJ is constant, and the shear yield stress is τ_Y.
Solve the governing equation for $\phi(z)$ and determine the value of t_0 at which the bar first yields.



Solution

The governing equation and fixed-end conditions are

$$GJ\phi'' + t_0 z = 0, \qquad \phi(0) = \phi(L) = 0.$$

Integrating twice gives

$$\phi(z) = -\frac{t_0 z^3}{6GJ} + C_1 z + C_2.$$

The boundary conditions give $C_2 = 0$ and $C_1 = t_0 L^2/(6GJ)$, so

$$\boxed{\phi(z) = \frac{t_0}{6GJ} z(L^2 - z^2)}, \qquad \boxed{T(z) = GJ\phi'(z) = \frac{t_0}{6}(L^2 - 3z^2).}$$

At the two ends,

$$T(0) = \frac{t_0 L^2}{6}, \qquad T(L) = -\frac{t_0 L^2}{3}.$$

The largest torque magnitude occurs at $z = L$. The largest shear stress occurs there at the outer surface:

$$\tau_{\max} = \frac{|T(L)|R}{J} = \frac{t_0 L^2 R}{3J}.$$

Setting this equal to τ_Y gives

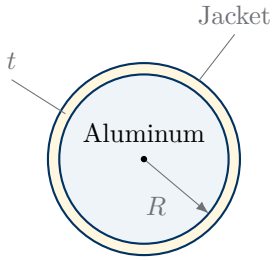
$$\boxed{t_{0,Y} = \frac{3J\tau_Y}{L^2 R} = \frac{3\pi R^3 \tau_Y}{2L^2}.}$$

Although both ends have zero rotation, they carry nonzero reaction torques.

Example 4: Choosing a lightweight jacket

A composite torsion bar of length L has an aluminum core of radius R and a bonded thin jacket of thickness t. It must carry a prescribed ultimate torque T_u with minimum weight.
Choose the jacket material from the table below and determine the required thickness. Assume $t \ll R$. For this estimate, use the fully plastic torque of the aluminum core and take the jacket stress to be its tabulated limit τ_Y. The prescribed torque satisfies

$$T_u > \frac{2\pi}{3} R^3 \tau_Y^{\mathrm{Al}}.$$



Material	G [GPa]	τ_Y [MPa]	Density [kg/m^3]
Aluminum	28.5	275	2700
Mild steel	80	154	7850
HS steel	80	770	7850
HTS graphite–5208 epoxy	61	740	1550
Alumina (Al_2O_3)	161	9350	4000

Material data from [G], Table 7.3.

Hint: First express T_u in terms of the core and jacket capacities. Solve for t, then compare the resulting masses.

Solution

For the fully plastic core, the shear stress has magnitude τ_Y^{Al} throughout the cross-section. Thus,

$$T_{\mathrm{core}} = \int_0^R \rho \tau_Y^{\mathrm{Al}} (2\pi\rho\, d\rho) = \frac{2\pi}{3} R^3 \tau_Y^{\mathrm{Al}}.$$

For the thin jacket, its area is approximately $2\pi Rt$ and its moment arm is approximately R. At its limiting stress,

$$T_{\mathrm{jacket}} \approx 2\pi R^2 t \tau_Y^{(j)}.$$

Adding the capacities and solving for thickness gives

$$T_u \approx T_{\mathrm{core}} + 2\pi R^2 t \tau_Y^{(j)}, \qquad \boxed{t \approx \frac{T_u - T_{\mathrm{core}}}{2\pi R^2 \tau_Y^{(j)}}}.$$

Let ρ_{Al} and ρ_j denote the mass densities. Substituting the thickness into the bar's mass gives

$$m \approx \rho_{\mathrm{Al}} \pi R^2 L + \rho_j (2\pi R t) L = \rho_{\mathrm{Al}} \pi R^2 L + \frac{L(T_u - T_{\mathrm{core}})}{R} \frac{\rho_j}{\tau_Y^{(j)}}.$$

Only the last ratio depends on the jacket material. Alumina has the smallest value:

$$\frac{\rho_j}{\tau_Y^{(j)}} = \frac{4000}{9350} \approx 0.428 \, \frac{\mathrm{kg/m^3}}{\mathrm{MPa}}.$$

Therefore, within the stated model,

$$\boxed{\text{Choose alumina } (Al_2O_3).}$$

Its required thickness follows from the expression above with $\tau_Y^{(j)} = 9350\,\mathrm{MPa}$. The resulting t must satisfy the assumed $t \ll R$.