CE 30 / ME 85 — Introduction to Solid Mechanics Fall 2026

# Discussion 10

Weekly Material Recap and Example Problems

**Instructor:** Tim Carlson
**Contact:** timothy.carlson@berkeley.edu

**Date:** November 5, 2026
**Version:** Instructor copy

## 1 General reminders

- **Assignment reminder:** Homework 10 is due Friday, November 6, at 5:59 pm. This week's reading is [G], Sections 8.1–8.3.
- **Midterm 2 regrades:** Requests were to be submitted through Gradescope during the 24-hour regrade window.

## 2 October 26–October 30: material recap

### 2.1 Bending kinematics

We consider slender beams whose cross-sections have a vertical line of symmetry in both geometry and material, with loading in that plane. We assume small strains and rotations.

The central assumption is that **plane cross-sections remain plane and perpendicular to the deformed beam axis**. A section can translate and rotate, but it does not warp.

**Bending strain**

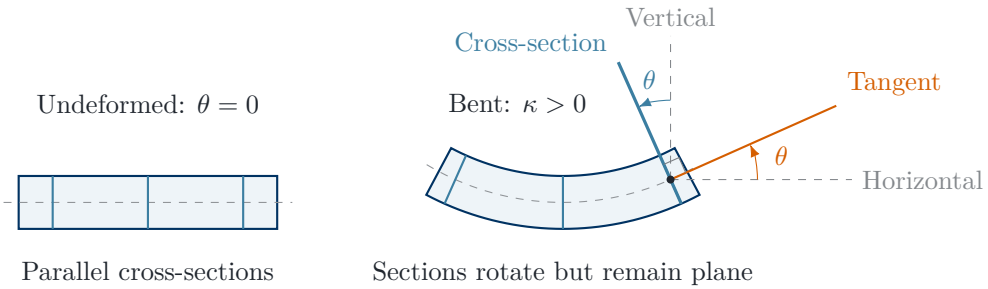
Let $v(x)$ be the vertical displacement of the beam axis and $\theta(x)$ the rotation of a cross-section from its original vertical orientation. Then

$$\theta = \frac{dv}{dx}, \qquad \kappa = \frac{d\theta}{dx} = \frac{d^2v}{dx^2}.$$

At height $y$, the rotation gives an axial displacement $u = -y\theta$. Differentiating along the beam gives

$$\boxed{\epsilon_{xx} = -y\kappa.}$$

Here, $y$ is measured upward from the **neutral axis**, where the axial strain is zero. We will determine its location from the condition of zero resultant axial force.




The two angles are equal because each cross-section stays perpendicular to the beam axis.

### 2.2 Locating the neutral axis

Combining the bending strain with linear elasticity gives $\sigma_{xx} = E\epsilon_{xx} = -E\kappa y$. The section resultants are

$$R = \int_A \sigma_{xx}\, dA, \qquad M = -\int_A y\sigma_{xx}\, dA.$$

We consider bending with $R = 0$. Our moment convention gives compression above the neutral axis when $M > 0$.

**Neutral axis for a linear elastic section**

Let $y_b$ measure height from the bottom of the section, and let $y_{\text{NA}}$ be the height of the neutral axis. Since $y = y_b - y_{\text{NA}}$,

$$0 = R = -\kappa \int_A E(y_b - y_{\text{NA}})\, dA.$$

Thus, the neutral axis passes through the $E$-weighted centroid:

$$\boxed{y_{\text{NA}} = \frac{\int_A E y_b\, dA}{\int_A E\, dA} = \frac{\sum_i E_i A_i y_{c,i}}{\sum_i E_i A_i}.}$$

The sum applies to pieces with constant $E_i$; $y_{c,i}$ is the height of each piece's centroid. For a homogeneous section, $E$ cancels and the neutral axis passes through the geometric centroid.

### 2.3 Bending stiffness and stress

Substituting the stress into the moment definition gives

$$M = -\int_A y(-E\kappa y)\, dA = \kappa \int_A E y^2\, dA.$$

**From bending moment to stress**

For a bonded, linear elastic composite section,

$$\boxed{(EI)_{\text{eff}} = \sum_i E_i \left[ I_{c,i} + A_i (y_{c,i} - y_{\text{NA}})^2 \right],}$$

where $I_{c,i}$ is each piece's second moment of area about its own horizontal centroidal axis. For a rectangle of width $b$ and height $h$, $I_c = bh^3/12$.

$$\boxed{\kappa = \frac{M}{(EI)_{\text{eff}}},} \qquad \boxed{\sigma_{xx}^{(i)} = -E_i \kappa (y_b - y_{\text{NA}}).}$$

For a homogeneous section, these reduce to

$$M = EI\kappa, \qquad \boxed{\sigma_{xx} = -\frac{My}{I},} \qquad I = \int_A y^2\, dA.$$

**Bonded materials share the same strain at a given height.** Strain varies linearly through the depth, but axial stress can jump at an interface because the moduli differ.

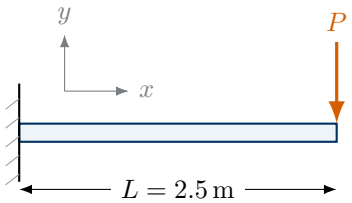
## 3 Example problems

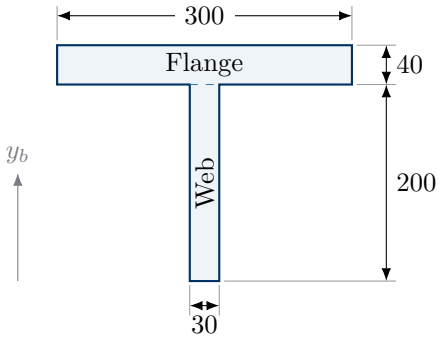
Use consistent units and take tension as positive.

### Example 1: Bending stress in a homogeneous T-section

A cantilever of length $L = 2.5\,\text{m}$ carries a downward force $P = 4\,\text{kN}$ at its free end. The entire section is made from one linear elastic material with $E = 10\,\text{GPa}$.
The flange is 300 mm wide and 40 mm thick. The web is 30 mm thick and 200 mm high, measured below the flange. Measure $y_b$ upward from the bottom.

1. Locate the neutral axis and calculate $I$ about it.
2. At the fixed end, determine the bending moment, curvature, and largest tensile and compressive stresses. Sketch the stress through the depth.




Cross-section dimensions in mm.

### Solution

**1. Neutral axis and second moment of area.**
Split the section into its flange and web:

| Part | $A_i$ [mm$^2$] | $y_{c,i}$ [mm] |
|---|---|---|
| Flange | $300(40) = 12000$ | 220 |
| Web | $30(200) = 6000$ | 100 |

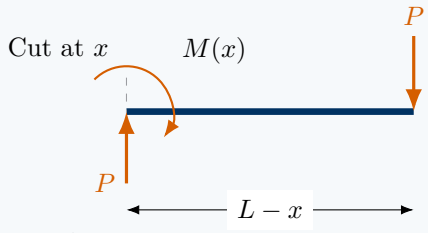
For one material,

$$\boxed{y_{\text{NA}} = \frac{12000(220) + 6000(100)}{12000 + 6000} = 180\,\text{mm}.}$$

Using the parallel-axis theorem,

$$\begin{aligned} I &= \left[\frac{300(40)^3}{12} + 12000(220 - 180)^2\right] \\ &\quad + \left[\frac{30(200)^3}{12} + 6000(100 - 180)^2\right] \\ &= \boxed{79.2 \times 10^6\,\text{mm}^4.} \end{aligned}$$

**2. Moment, curvature, and stress.**
Cut at $x$ and isolate the right portion. On this face, positive $M(x)$ acts clockwise.




Moment equilibrium about the cut gives

$$-M(x) - P(L - x) = 0.$$

Thus, at the fixed end,

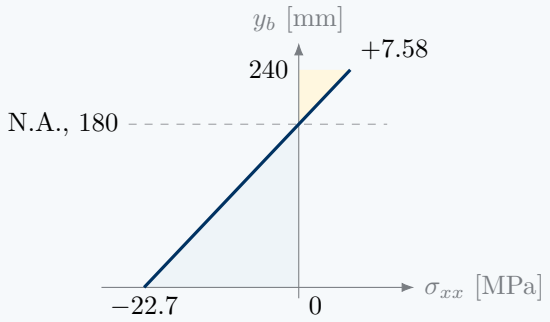
$$\boxed{M(0) = -PL = -10\,\text{kN m} = -10^7\,\text{N mm}.}$$

With $E = 10^4\,\text{N/mm}^2$,

$$\boxed{\kappa = \frac{M}{EI} = -1.263 \times 10^{-5}\,\text{mm}^{-1}.}$$

The top and bottom are 60 mm above and 180 mm below the neutral axis:

$$\boxed{\sigma_{\text{top}} = +7.58\,\text{MPa} \quad \text{(tension)}}, \qquad \boxed{\sigma_{\text{bottom}} = -22.7\,\text{MPa} \quad \text{(compression)}}.$$



## Example 2: The same section with two materials

Keep the geometry and loading from Example 1. The flange now has $E_f = 10\,\text{GPa}$ and the web has $E_w = 20\,\text{GPa}$. The materials are perfectly bonded and remain linear elastic.

1. Find $y_{\text{NA}}$ and $(EI)_{\text{eff}}$. Explain why the neutral axis moves.
2. At the fixed end, determine the curvature and the stress at the top, bottom, and both sides of the flange–web joint. Sketch the strain and stress through the depth.

## Solution

**1. Neutral axis and bending stiffness.**
Weighting the areas and centroid locations from Example 1 by the moduli,

$$\boxed{y_{\text{NA}} = \frac{10(12000)(220) + 20(6000)(100)}{10(12000) + 20(6000)} = 160\,\text{mm}.}$$

The neutral axis moves toward the stiffer web. About this new axis,

$$I_f = \frac{300(40)^3}{12} + 12000(220 - 160)^2 = 44.8 \times 10^6\,\text{mm}^4,$$
$$I_w = \frac{30(200)^3}{12} + 6000(100 - 160)^2 = 41.6 \times 10^6\,\text{mm}^4.$$

$$\boxed{(EI)_{\text{eff}} = E_f I_f + E_w I_w = 1.28 \times 10^{12}\,\text{N mm}^2.}$$

**2. Curvature, strain, and stress.**
The loading still gives $M(0) = -10^7\,\text{N mm}$, so

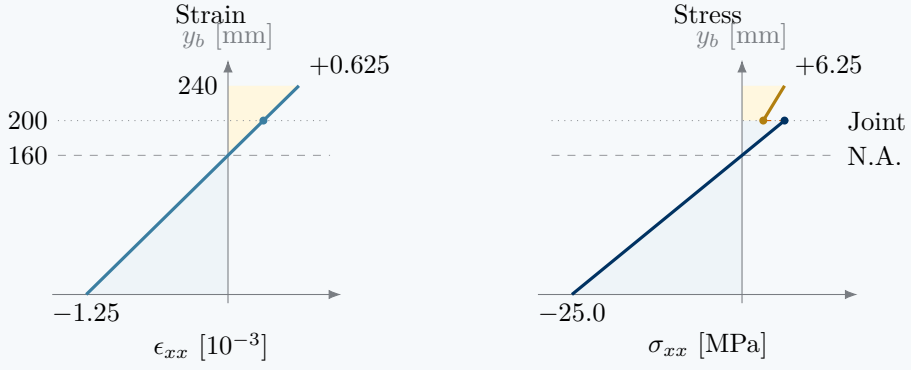
$$\boxed{\kappa = -7.8125 \times 10^{-6}\,\text{mm}^{-1}.}$$

At each height, use

$$\epsilon_{xx} = -\kappa(y_b - 160\,\text{mm}), \qquad \sigma_{xx} = E_i \epsilon_{xx}.$$

| Location | $y_b$ [mm] | $\epsilon_{xx}$ | $\sigma_{xx}$ [MPa] |
|---|---|---|---|
| Top of flange | 240 | $+6.25 \times 10^{-4}$ | $+6.25$ |
| Joint, flange side | 200 | $+3.125 \times 10^{-4}$ | $+3.125$ |
| Joint, web side | 200 | $+3.125 \times 10^{-4}$ | $+6.25$ |
| Bottom of web | 0 | $-1.25 \times 10^{-3}$ | $-25.0$ |

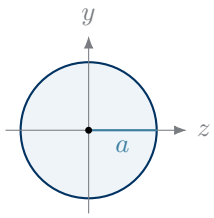
At the joint, strain is continuous. The web has twice the flange's modulus, so its axial stress is twice as large.



## Example 3: A nonlinear material in bending

A solid circular beam of radius $a$ is in pure bending with curvature $\kappa > 0$ and no resultant axial force. Its material obeys

$$\sigma_{xx} = C_1\epsilon_{xx} + C_2\epsilon_{xx}^3, \qquad C_1, C_2 > 0.$$




Determine $M(\kappa)$. Does doubling the curvature double the moment?

You may use the following integrals about a diameter:

$$\int_A y^2\,dA = \frac{\pi a^4}{4}, \qquad \int_A y^4\,dA = \frac{\pi a^6}{8}.$$

## Solution

The circular section is symmetric, and the constitutive law gives equal and opposite stresses for equal and opposite strains. Thus, $R = 0$ when the neutral axis passes through the center.
The bending strain still gives

$$\epsilon_{xx} = -\kappa y, \qquad \sigma_{xx} = -C_1\kappa y - C_2\kappa^3 y^3.$$

Using the definition of the moment,

$$\begin{aligned} M &= -\int_A y\sigma_{xx}\,dA \\ &= C_1\kappa\int_A y^2\,dA + C_2\kappa^3\int_A y^4\,dA \\ &= \boxed{\frac{\pi a^4 C_1}{4}\kappa + \frac{\pi a^6 C_2}{8}\kappa^3.} \end{aligned}$$

Doubling the curvature doubles the linear contribution but makes the cubic contribution eight times as large. The moment therefore increases by more than a factor of two.