CE 30 / ME 85 — Introduction to Solid Mechanics **Fall 2026**

Discussion 11

Weekly Material Recap and Example Problems

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1 General reminders

- **Schedule update:** Lectures have been shifted by 1 (updated syllabus now on bcourses).
- **Thanksgiving:** Thursday office hour and discussion canceled, enjoy the break! Monday the 23rd TBD.

2 November 2–November 6: material recap

2.1 Beam deflections and boundary conditions

Last week, we related bending moment to curvature, which can be used to solve for vertical deflection. Indeed, for the small deformations and linear elastic materials relevant to this course, the derivative of the vertical deflection is the rotation, and the derivative of the rotation is the curvature.

From loading to deflection

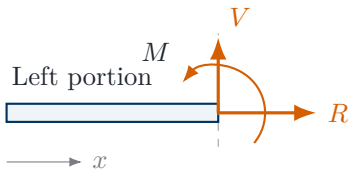
Take v and the distributed load q as positive upward. In the absence of thermal strains,

$$\theta = v', \qquad M = EIv'', \qquad V = -(EIv'')'.$$

Combining these with equilibrium gives

$$\boxed{(EIv'')'' = q.} \qquad \text{For constant } EI: \quad EIv'''' = q.$$

Here, $'$ denotes derivatives with respect to x. Note that solving a fourth-order ODE requires the specification of four boundary conditions.



Positive internal resultants on the right cut face of the left portion, following [G], Chapter 8. Positive σ_{xx} denotes tension.

Possible boundary conditions on v:

End condition	Conditions on deflection
Fixed	$v = 0, \quad v' = 0$
Pin or roller, with no applied end couple	$v = 0, \quad EIv'' = 0$
Free, with no end force or couple	$EIv'' = 0, \quad (EIv'')' = 0$
Vertical guide: slides freely, cannot rotate	$v' = 0, \quad (EIv'')' = 0$
Vertical spring k at the right end, pin-connected	$EIv''(L) = 0, \quad (EIv'')'(L) = kv(L)$

2.2 Biaxial bending

A transverse load can bend a beam in both the x–y and x–z planes. Let v and w denote displacements in the y and z directions. We use the book's curvature convention:

$$\kappa_z = v'', \qquad \kappa_y = -w'', \qquad \epsilon_{xx} = -\kappa_z y + \kappa_y z.$$

Add the two linear elastic bending responses

For a homogeneous section, choose y, z through the centroid along symmetry axes, so that $\int_A yz\, dA = 0$. With no resultant axial force,

$$M_z = EI_z\kappa_z, \qquad M_y = EI_y\kappa_y, \qquad I_z = \int_A y^2\, dA, \quad I_y = \int_A z^2\, dA.$$

Thus,

$$\boxed{\sigma_{xx}(x, y, z) = -\frac{M_z(x)y}{I_z} + \frac{M_y(x)z}{I_y}}, \qquad v'' = \frac{M_z}{EI_z}, \quad w'' = -\frac{M_y}{EI_y}.$$

Superposition requires the **linear elastic, small-deflection** model. Add the stress contributions at the **same point** (y, z).

The neutral axis is the line where $\sigma_{xx} = 0$; it need not coincide with either coordinate axis. Also, I_z measures resistance to bending *about* z, even though its integrand contains y^2.

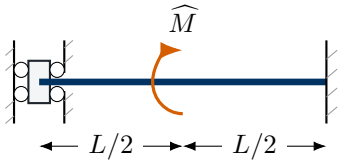
3 Example problems

Example 1: Choosing the loading and boundary conditions

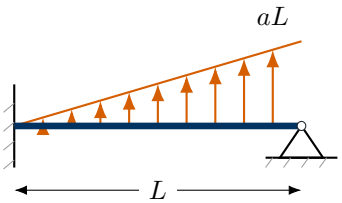
Both beams have length L and constant bending stiffness EI. For each, write $q(x)$ and four boundary conditions in terms of v and its derivatives.

(a) The left guide permits vertical motion but prevents rotation; the right end is fixed. A clockwise couple $\widehat{M}$ acts at $x = L/2$. Also state the jump in internal moment there.

(b) The left end is fixed and the right end is pinned. The upward load increases linearly from zero to aL, where $a > 0$ has units of force per length squared. Solve for the deflection and plot V, M, θ, and v.



(a) [G], Exercise 8.25(c)



(b) [G], Exercise 8.26(a)

Measure x from the left end in both cases.

Solution

(a) Guided–fixed beam with an interior couple.
The guide prevents rotation but supplies no vertical force. The fixed end prevents both translation and rotation:

$$\boxed{v'(0) = 0, \quad EIv'''(0) = 0, \quad v(L) = 0, \quad v'(L) = 0.}$$

A clockwise point couple is represented in the beam equation by the derivative of a delta:

$$\boxed{q(x) = \widehat{M}\,\delta'(x - L/2).}$$

In a piecewise solution, use $q = 0$ on each side of the couple and match deflection, slope, and shear across it. The internal moment jumps:

$$\boxed{M((L/2)^+) - M((L/2)^-) = \widehat{M}.}$$

The sign follows directly from moment equilibrium of a short piece around the couple: $-M^- + M^+ - \widehat{M} = 0$.

(b) Fixed–pinned beam with a triangular load.
The intensity is positive upward and has slope a, so

$$\boxed{q(x) = ax, \qquad 0 < x < L.}$$

The fixed end prevents translation and rotation. The pin prevents vertical translation and supplies no end couple:

$$\boxed{v(0) = 0, \quad v'(0) = 0, \quad v(L) = 0, \quad EIv''(L) = 0.}$$

The pin can supply a vertical reaction, so $v'''(L)$ is not prescribed to be zero.

Integrate before applying the conditions. Since EI is constant, four integrations give

$$\begin{aligned}
EIv'''' &= ax, \\
EIv''' &= \frac{ax^2}{2} + C_1, \\
EIv'' &= \frac{ax^3}{6} + C_1x + C_2, \\
EIv' &= \frac{ax^4}{24} + \frac{C_1x^2}{2} + C_2x + C_3, \\
EIv &= \frac{ax^5}{120} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4.
\end{aligned}$$

The two fixed-end conditions immediately give

$$v'(0) = 0 \implies C_3 = 0, \qquad v(0) = 0 \implies C_4 = 0.$$

The remaining constants encode the unknown internal resultants at the left end: $V(0) = -C_1$ and $M(0) = C_2$.

Use the pin conditions to solve a 2×2 system. At $x = L$,

$$\begin{aligned}
M(L) = 0: &\quad \frac{aL^3}{6} + C_1L + C_2 = 0, \\
v(L) = 0: &\quad \frac{aL^5}{120} + \frac{C_1L^3}{6} + \frac{C_2L^2}{2} = 0.
\end{aligned}$$

Multiplying the second equation by $6/L^2$ gives

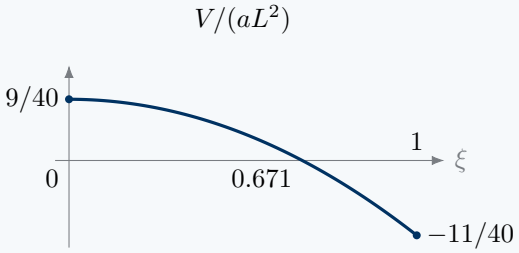
$$\boxed{\begin{pmatrix} L & 1 \\ L & 3 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = -aL^3 \begin{pmatrix} 1/6 \\ 1/20 \end{pmatrix}.}$$

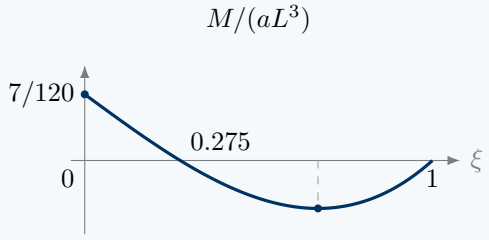
Subtracting the first row from the second gives $2C_2 = 7aL^3/60$. Substitution into the first row then yields

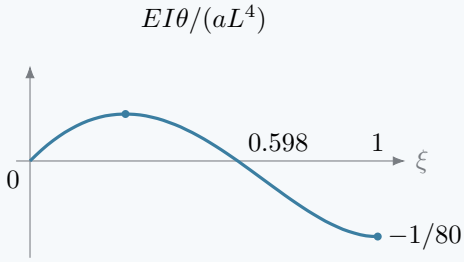
$$\boxed{C_1 = -\frac{9aL^2}{40}, \qquad C_2 = \frac{7aL^3}{120}.}$$

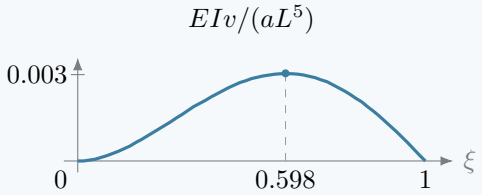
Recover the fields and draw the diagrams.
To plot without choosing numerical values of a, L, E, I, let $\xi = x/L$. Using $V = -EIv'''$:

$$\begin{aligned} V(x) &= aL^2 \left(\frac{9}{40} - \frac{\xi^2}{2} \right), \\ M(x) &= aL^3 \left(\frac{\xi^3}{6} - \frac{9\xi}{40} + \frac{7}{120} \right), \\ \theta(x) &= \frac{aL^4}{EI} \left(\frac{\xi^4}{24} - \frac{9\xi^2}{80} + \frac{7\xi}{120} \right), \\ v(x) &= \frac{aL^5}{EI} \left(\frac{\xi^5}{120} - \frac{3\xi^3}{80} + \frac{7\xi^2}{240} \right). \end{aligned}$$









Extrema follow $M' = -V$, $\theta' = M/(EI)$, and $v' = \theta$. The maximum upward deflection occurs at $x \approx 0.598L$, where $\theta = 0$. At the pin, $v(L) = M(L) = 0$, but neither the slope nor the shear is zero.

Example 2: An oblique load on a rectangular cantilever

A homogeneous, linear elastic cantilever has length L, modulus E, and a rectangular section of height $2b$ and width b. A tip force $\boldsymbol{P} = -P\boldsymbol{e}_y + P\boldsymbol{e}_z$, with $P > 0$, has the two components shown below. Section coordinates are measured from the centroid.

1. Find I_y, I_z and the fixed-end stress $\sigma_{xx}(0, y, z)$. Locate the neutral axis and the largest tensile and compressive stresses.
2. Find the tip displacements $v(L)$ and $w(L)$. Does the tip move in the direction of the applied force?



Solution

1. Stress and neutral axis.

$$I_z = \frac{b(2b)^3}{12} = \frac{2b^4}{3}, \qquad I_y = \frac{(2b)b^3}{12} = \frac{b^4}{6}.$$

On the right portion, the cut moment is $-\boldsymbol{M}(x)$. Equilibrium gives

$$\boldsymbol{M}(x) = (L - x)\boldsymbol{e}_x \times (-P\boldsymbol{e}_y + P\boldsymbol{e}_z) = -P(L - x)(\boldsymbol{e}_y + \boldsymbol{e}_z).$$

At the fixed end,

$$\boxed{\sigma_{xx}(0, y, z) = \frac{3PL}{2b^4}y - \frac{6PL}{b^4}z.}$$

The neutral axis is $\boxed{y = 4z}$. The largest stresses occur at opposite corners:

$$\sigma(0, b, -b/2) = +\frac{9PL}{2b^3}, \qquad \sigma(0, -b, b/2) = -\frac{9PL}{2b^3}.$$

2. Tip displacement.
Integrate $v'' = M_z/(EI_z)$ and $w'' = -M_y/(EI_y)$ twice, using zero displacement and slope at $x = 0$:

$$v(x) = -\frac{Px^2(3L - x)}{6EI_z}, \qquad w(x) = \frac{Px^2(3L - x)}{6EI_y}.$$

At the tip,

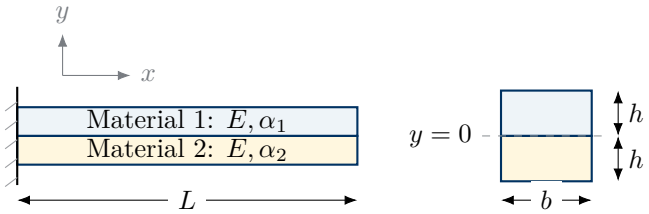
$$\boxed{v(L) = -\frac{PL^3}{2Eb^4}}, \qquad \boxed{w(L) = \frac{2PL^3}{Eb^4}}.$$

Thus $|w(L)| = 4|v(L)|$: the tip does not move parallel to the force. The beam is four times as stiff in the y direction as in the z direction.

4 Challenge exercise

Example 3: A thermoelastic switch

Two layers are perfectly bonded to form a cantilever of length L and width b. Each layer has thickness h and Young's modulus E, but their thermal expansion coefficients are α_1 and α_2. Material 1 is on top. The beam is initially straight and stress-free.
The temperature changes uniformly by ΔT, with no applied mechanical loads. Find the vertical tip deflection $v(L)$. For heating with $\alpha_1 > \alpha_2$, does the tip move up or down?



Hint: With $y = 0$ at the interface, allow both extension and curvature: $\epsilon_{xx} = \epsilon_0 - \kappa y$ for $\epsilon_0 = \epsilon_{xx}(y = 0)$. In material i, $\sigma_{xx} = E(\epsilon_{xx} - \alpha_i \Delta T)$.
Adapted from [G], Exercise 8.35.

Solution

With no applied mechanical loads, each section has $R = M = 0$. Axial equilibrium gives

$$\begin{aligned} 0 = R &= Eb\left[\int_{-h}^{0} (\epsilon_0 - \kappa y - \alpha_2 \Delta T)\, dy + \int_{0}^{h} (\epsilon_0 - \kappa y - \alpha_1 \Delta T)\, dy\right] \\ &= Ebh\left[2\epsilon_0 - (\alpha_1 + \alpha_2)\Delta T\right]. \end{aligned}$$

Hence $\epsilon_0 = (\alpha_1 + \alpha_2)\Delta T/2$. Using the same stresses in $M = -\int_A y\sigma_{xx}\, dA$ gives

$$0 = M = Eb\left[\frac{2h^3}{3}\kappa + \frac{h^2}{2}(\alpha_1 - \alpha_2)\Delta T\right].$$

Therefore, the constant curvature is

$$\boxed{\kappa = \frac{3(\alpha_2 - \alpha_1)\Delta T}{4h}.}$$

Integrate $v'' = \kappa$ using the fixed-end conditions $v(0) = v'(0) = 0$:

$$v(x) = \frac{\kappa x^2}{2}, \qquad \boxed{v(L) = \frac{3(\alpha_2 - \alpha_1)\Delta T L^2}{8h}.}$$

If $\alpha_1 > \alpha_2$ and $\Delta T > 0$, the tip moves **downward**. The top layer wants to expand more and lies on the outside of the bend.
Observe that in the setting of thermo-elasticity, curvature can exist without any net moment!